KAM Theorem for Gevrey Hamiltonians

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Abstract

We consider Gevrey perturbations H of a completely integrable Gevrey Hamiltonian H_0. Given a Cantor set Ω_κ defined by a Diophantine condition, we find a family of KAM invariant tori of H with frequencies $\omega \in \Omega_\kappa$ which is Gevrey smooth in a Whitney sense. Moreover, we obtain a symplectic Gevrey normal form of the Hamiltonian in a neighborhood of the union Λ of the invariant tori. This leads to effective stability of the quasiperiodic motion near Λ.

1 *KAM theorem for Gevrey Hamiltonians*

Let D^0 be a bounded domain in $\mathbf{R}^n$, and $\mathbf{T}^n = \mathbf{R}^n/2\pi\mathbf{Z}^n$, $n \geq 2$. We consider a class of real valued Gevrey Hamiltonians in $\mathbf{T}^n \times D^0$ which are small perturbations of a real valued non-degenerate Gevrey Hamiltonian $H^0(I)$ depending only on the action variables $I \in D^0$. Our aim is to obtain a family of KAM (Kolmogorov-Arnold-Moser) invariant tori Λ_ω of H with frequencies ω in a suitable Cantor set Ω_κ defined by a Diophantine condition and to prove Gevrey regularity for it. It turns out that for each $\omega \in \Omega_\kappa$, Λ_ω is a Gevrey smooth embedded torus having the same Gevrey regularity as the Hamiltonian H. Moreover, we shall prove that the family Λ_ω, $\omega \in \Omega_\kappa$, is Gevrey smooth with respect to ω in a Whitney sense, with a Gevrey index depending on the Gevrey class of H and on the exponent in the Diophantine condition. This naturally involves anisotropic Gevrey classes. Let $\rho_1, \rho_2 \geq 1$ and L_1, L_2 be positive constants. Given a domain $D \subset \mathbf{R}^n$, we denote by $\mathcal{G}^{\rho_1,\rho_2}_{L_1,L_2}(\mathbf{T}^n \times D)$ the set of all C^∞ real valued Hamiltonians H in $\mathbf{T}^n \times D$ such that

$$\|H\|_{L_1,L_2} := \sup_{\alpha,\beta\in\mathbf{N}^n} \ \sup_{(\theta,I)\in\mathbf{T}^n\times D^0} \left(|\partial_\theta^\alpha \partial_I^\beta H(\theta,I)|\, L_1^{-|\alpha|} L_2^{-|\beta|} \alpha!^{-\rho_1} \beta!^{-\rho_2} \right) \ < \ \infty\,, \tag{1.1}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$. In the same way we define $\mathcal{G}^{\rho_1,\rho_2}_{L_1,L_2}(\mathbf{T}^n \times \overline{D})$, where $\overline{D}$ is the closure of D. If $\rho_1 = \rho_2 = \rho$ we write also $\mathcal{G}^{\rho}_{L_1,L_2}(\mathbf{T}^n \times D)$, and sometimes we do not indicate the Gevrey constants L_1, L_2.

Let H^0 be a completely integrable real valued Gevrey smooth Hamiltonian $\mathbf{T}^n \times D^0 \ni (\theta, I) \to H^0(I) \in \mathbf{R}$. We suppose that H^0 is non-degenerate, which means that the map $\nabla H^0 : D^0 \to \Omega^0$ is a diffeomorphism. Denote by $g^0 \in C^\infty(\Omega^0)$ the Legendre transform of H^0 (then $\nabla g^0 : \Omega^0 \to D^0$ is the inverse map to ∇H^0). We suppose also that there are positive constants $\rho > 1$, $A_0 > 0$, and $L_0 \leq L_2$ such that $H^0 \in \mathcal{G}^\rho_{L_0}(D^0)$, $g^0 \in \mathcal{G}^\rho_{L_0}(\Omega^0)$, and

$$\|H^0\|_{L_0}\,,\ \|g^0\|_{L_0} \ \leq \ A_0 \tag{1.2}$$

in the corresponding norms, defined as in (1.1). In particular, Ω^0 is a bounded domain. Given a subdomain D of D^0 we set $\Omega := \nabla H^0(D) \subset \Omega^0$. Fix $\tau > n-1$ and $\kappa > 0$. We denote by Ω_κ

the set of all frequencies $\omega \in \Omega$ having distance $\geq \kappa$ to the boundary of Ω and also satisfying the Diophantine condition

$$|\langle \omega, k \rangle| \ \geq \ \frac{\kappa}{|k|^\tau}\,, \quad \text{for all } 0 \neq k \in \mathbf{Z}^n\,, \tag{1.3}$$

where $|k| = |k_1| + \cdots + |k_n|$.

We are going to find a Gevrey family of KAM invariant tori with frequencies in Ω_κ for small perturbations of H^0 in $\mathcal{G}^\rho_{L_1,L_2}$. In what follows we fix the constants A_0 and L_0, and allow the constants $L_2 \geq L_1 \geq 1$ to be arbitrary large. This occurs in the case of the elliptic equilibrium for example ($L_2 \gg 1$). Given $\omega \in \Omega$, we denote by $\mathcal{L}_\omega = \langle \omega, \partial_\varphi \rangle = \sum_{j=1}^n \omega_j \partial/\partial\varphi_j$ the corresponding vectorfield on $\mathbf{T}^n$. Fix $0 < \varsigma \leq 1$.

Theorem 1.1 *Let H^0 be a real valued non-degenerate $\mathcal{G}^\rho$-smooth Hamiltonian , $\rho > 1$, depending only on $I \in D^0$ and satisfying (1.2). Let D be a subdomain of D^0 with $\overline{D} \subset D^0$, and $\Omega = \nabla H^0(D)$. Fix $L_2 \geq L_1 \geq 1$ and $\kappa \leq L_2^{-1-\varsigma}$ such that $L_2 \geq L_0$ and $\Omega_\kappa \neq \emptyset$. Then there exists $N = N(n,\rho,\tau) > 0$ and $\epsilon > 0$ independent of κ, L_1, L_2, and of the domain $D \subset D^0$, such that for any $H \in \mathcal{G}^\rho_{L_1,L_2}(\mathbf{T}^n \times \overline{D})$ with norm*

$$\epsilon_H := \kappa^{-2} \|H - H^0\|_{L_1,L_2} \leq \epsilon L_1^{-N},$$

there exists a map $\overline{\Phi} := (\overline{U}, \overline{V}) : \mathbf{T}^n \times \Omega \to D$ of an anisotropic Gevrey class $\mathcal{G}^{\rho,\rho'}$, $\rho' = \rho(\tau+1)+1$, such that

(i) *For each $\omega \in \Omega_\kappa$, $\Lambda_\omega := \{(\overline{\Phi}(\theta,\omega)) : \theta \in \mathbf{T}^n\}$ is an embedded Lagrangian invariant torus of H and $X_H \circ \overline{\Phi}(\cdot,\omega) = D\overline{\Phi}(\cdot,\omega) \cdot \mathcal{L}_\omega$.*

(ii) *There are constants $A, C > 0$, independent of κ, L_1, L_2, and of D, such that*

$$\left|\partial_\theta^\alpha \partial_\omega^\beta (\overline{U}(\theta;\omega) - \theta)\right| + \kappa^{-1} \left|\partial_\theta^\alpha \partial_\omega^\beta (\overline{V}(\theta;\omega) - \nabla g^0(\omega))\right|$$

$$\leq \ A\, C_1^{|\alpha|} \left(C_2 \kappa^{-1}\right)^{|\beta|} \alpha!^{\,\rho} \beta!^{\,\rho(\tau+1)+1} L_1^{N/2} \sqrt{\epsilon_H}\,,$$

uniformly in $(\theta,\omega) \in \mathbf{T}^n \times \Omega$ and for any $\alpha, \beta \in \mathbf{N}^n$, where $C_1 = CL_1$ and $C_2 = CL_1^{\tau+1}$.

Note that $\overline{\Phi}$ belongs to $\mathcal{G}^{\rho,\rho'}_{\bar{L}_1,\bar{L}_2}(\mathbf{T}^n \times \overline{D})$ with Gevrey constants $\bar{L}_1 = CL_1$ and $\bar{L}_2 = C_2\kappa^{-1} \geq CL_1^{\tau+1}L_2^{1+\varsigma}$. As a consequence, we obtain a symplectic normal form of H near the union of the invariant tori. We say that a real valued function $\Phi \in C^\infty(\mathbf{R}^n \times D)$ is a generating function of an exact symplectic map $\chi : \mathbf{T}^n \times D \to \mathbf{T}^n \times D$ if $\Phi(x,I) - \langle x, I \rangle$ is 2π-periodic with respect to x, $|\mathrm{Id} - \Phi_I| < 1$, and

$$\{(\varphi, I; \chi(\varphi, I)) :\, (\varphi, I) \in \mathbf{T}^n \times D\} = \{(p(\Phi_I(x,I)), I; p(x), \Phi_x(x,I)) :\, (x,I) \in \mathbf{R}^n \times D\},$$

where $p : \mathbf{R}^n \to \mathbf{T}^n$ is the natural projection. Fix $\widetilde{\kappa} = \widetilde{\kappa}(D)$ so that the Lebesgue measure of $\Omega_{\widetilde{\kappa}}$ is positive. Define $\widetilde{\Omega}_\kappa$, $0 < \kappa \leq \widetilde{\kappa}(D)$, to be the set of points of a positive Lebesgue density in Ω_κ. In other words, $\omega \in \widetilde{\Omega}_\kappa$ if for any neighborhood U of ω in Ω the Lebesgue measure of $U \cap \Omega_\kappa$ is positive. Obviously, $\widetilde{\Omega}_\kappa$ and Ω_κ have the same Lebesgue measure.

Corollary 1.2 *Suppose that the hypothesis of Theorem 1.1 hold and $0 < \kappa \le \widetilde{\kappa}$. Then there exists $N = N(n,\rho,\tau) > 0$ and $\epsilon > 0$ independent of κ, L_1, L_2, and D, such that for any $H \in \mathcal{G}^{\rho}_{L_1,L_2}(\mathbf{T}^n \times \overline{D})$ with $\varepsilon_H \le \epsilon L_1^{-N-2(\tau+2)}$ there is a $\mathcal{G}^{\rho'}$-diffeomorphism $\omega : D \to \Omega$, and an exact symplectic transformation $\chi \in \mathcal{G}^{\rho,\rho'}(\mathbf{T}^n \times D, \mathbf{T}^n \times D)$ defined by a generating function $\Phi(x,I) = \langle x, I\rangle + \phi(x,I)$, $\phi \in \mathcal{G}^{\rho,\rho'}(\mathbf{T}^n \times D)$, such that the transformed Hamiltonian $\tilde{H}(\varphi,I) := H(\chi(\varphi,I))$ belongs to $\mathcal{G}^{\rho,\rho'}(\mathbf{T}^n \times D)$ and for each $I \in \omega^{-1}(\widetilde{\Omega}_\kappa)$, $\mathbf{T}^n \times \{I\}$ is an invariant torus of $\tilde{H}$. The functions $K(I) := \tilde{H}(0,I)$ and $R(\varphi,I) := \tilde{H}(\varphi,I) - K(I)$ satisfy*

$$\forall\, \alpha \in \mathbf{N}^n \ , \ \forall\, (\varphi,I) \in \mathbf{T}^n \times \omega^{-1}(\widetilde{\Omega}_\kappa) \ , \quad \partial^\alpha \nabla K(I) \,=\, \partial^\alpha \omega(I)\,,\ \partial_I^\alpha R(\varphi,I) \,=\, 0\,.$$

Moreover, there exist $A, C > 0$ independent κ, L_1, L_2, and of the domain D, such that

$$\left|\partial_\varphi^\alpha \partial_I^\beta \phi(\varphi,I)\right| \,+\, \left|\partial_I^\beta(\omega(I) - \nabla H^0(I))\right| \,+\, \left|\partial_\varphi^\alpha \partial_I^\beta(\tilde{H}(\varphi,I) - H^0(I))\right|$$

$$\le\ A\kappa C_1^{|\alpha|}\left(C_2\kappa^{-1}\right)^{|\beta|}\ \alpha!^{\,\rho}\beta!^{\,\rho'}\, L_1^{N/2}\sqrt{\epsilon_H}\,,$$

uniformly with respect to $(\varphi,I) \in \mathbf{T}^n \times D$ and for any $\alpha,\beta \in \mathbf{N}^n$, where $C_1 = CL_1$ and $C_2 = CL_1^{\tau+1}$.

Denote by $\Omega \ni \omega \mapsto I(\omega) \in D$ the inverse map to the diffeomorphism $I \mapsto \omega(I)$. Then for each $\omega \in \widetilde{\Omega}_\kappa$, the restriction of the Hamiltonian flow of $\tilde{H}$ to the invariant torus $\mathbf{T}^n \times \{I(\omega)\}$ is given by $(t,\varphi,I) \mapsto (\varphi + t\nabla K(I), I)$, $I = I(\omega)$. Set $E_\kappa = \omega^{-1}(\widetilde{\Omega}_\kappa)$. Expanding $\partial_\varphi^\alpha \partial_I^\beta R(\varphi,I)$ in Taylor series at some $I_0 \in E_\kappa$ such that $|I_0 - I| = |E_\kappa - I| = \inf_{I' \in E_\kappa} |I' - I|$, we obtain for any $\alpha,\beta \in \mathbf{N}^n$ and $m \in \mathbf{N}$

$$|\partial_\varphi^\alpha \partial_I^\beta R(\varphi,I)| \ \le\ \kappa A\, C_1^{|\alpha|}\left(C_2\kappa^{-1}\right)^{|\beta|+m}\ \alpha!^{\,\rho}\beta!^{\,\rho'}\, m!^{\,\rho'-1}, \quad (\varphi,I) \in \mathbf{T}^n \times D\,,\ I \notin E_\kappa\,,$$

where the positive constants A, C_1, C_2 are as above. Using Stirling's formula we minimize the right-hand side with respect to $m \in \mathbf{N}$ which leads to

$$|\partial_\varphi^\alpha \partial_I^\beta R(\varphi,I)| \ \le\ \kappa A\, C_1^{|\alpha|}\left(C_2\kappa^{-1}\right)^{|\beta|}\ \alpha!^{\,\rho}\beta!^{\,\rho'}\ \exp\left(-(\kappa C_2^{-1}|E_\kappa - I|)^{-\frac{1}{\rho(\tau+1)}}\right) \tag{1.4}$$

for any $\alpha,\beta \in \mathbf{N}^n$ uniformly with respect to $(\varphi,I) \in \mathbf{T}^n \times D\,,\ I \notin E_\kappa$, where the constants A, C_1, C_2 are as above. These inequalities yield effective stability of the quasiperiodic motion near the invariant tori as in [10]. Effective stability of the action along all the trajectories for Gevrey smooth Hamiltonians has been obtained recently in in [7]. The importance of the Gevrey category for that kind of problems is indicated by Lochak [6]. Integrability over a Cantor set of tori for C^∞ Hamiltonians is obtained by Pöschel [8] and Lazutkin (see [5] for references).

Theorem 1.1 and Corollary 1.2 hold in the case of a non-degenerate elliptic equilibrium for Gevrey Hamiltonians as in [10]. Indeed, let us consider the Birkhoff normal form of the Hamiltonian, namely, $H(\theta,I) = H^0(I) + H^1(\theta,I)$, where $H^0(I) = \langle \alpha^0, I\rangle + \langle QI, I\rangle$, with $\det Q \ne 0$, and $H^1(\theta,I) = O(|I|^{5/2})$, (θ,I) being suitable polar symplectic coordinates. Here, H^1 is Gevrey smooth in $\mathbf{T}^n \times D_a$, $D_a = \{c_0 a \le I_j \le c_0^{-1} a \,:\, j = 1,\dots,n\}$, and $0 < a \le a_0$, where $0 < c_0 < 1$ is fixed. More precisely, $H^1 \in \mathcal{G}^\rho_{L_1,L_2}(\mathbf{T}^n \times D_a)$ with norm $\|H^1\|_{L_1,L_2} = O(a^{5/2})$, where $L_2 = C_0 a^{-1}$, and the positive constants L_1 and C_0 are fixed. Then Theorem 1.1 holds choosing $\kappa = \delta a^{1+\varsigma}$, $0 < \varsigma < 1/4$, $0 < \delta \le 1$, and for any $0 < a \le a_0 \ll 1$.

As in [10] the symplectic normal form in Corollary 1.2 can be used to obtain Gevrey quantum integrability over the corresponding family of invarint tori and to construct quasimodes with exponentially small discrepancy in the semi-classical limit for Schrödinger type operators with Gevrey coefficients. Similar results could be obtained for more general classes of non quasi-analytic Hamiltonians as well.

The idea of the proof of Theorem 1.1 is close to that of Theorem 1 in [10] (see also [4]). It follows from a KAM theorem for a family of Hamiltonians $P(\theta, I; \omega)$, where the frequencies ω are taken as independent parameters. Here we follow closely the exposition of Pöschel [9]. First we prove an approximation lemma for Gevrey Hamiltonians P with real valued analytic Hamiltonians P_j in suitable complex domains in Sect. 3.1. To obtain P_j we first construct suitable almost analytic extension of P and then we use Green's formula. In Sect. 3.2 we recall from Pöschel [9] the KAM step and in Sect. 3.3 we set the parameters and make the iterations. Finally, using a Whitney extension theorem due to Bruna [1], we complete the proof of the theorem. In Sect. 3.6 we consider the case of real analytic Hamiltonians and we improve certain results in [10]. We prove in the Appendix an anisotropic version of the implicit function theorem of Komatsu [3] in Gevrey classes.

2 *KAM theorem for Gevrey Hamiltonians with parameters*

Consider the Hamiltonian $H(\theta, z) = H^0(z) + H^1(\theta, z)$ in $\mathbf{T}^n \times D^0$. Expanding $H^0(z)$ near given $z_0 \in D \subset D^0$, we write

$$H^0(z) = H^0(z_0) + \langle \nabla_z H^0(z_0), I\rangle + \int_0^1 (1-t)\langle \nabla_z^2 H^0(z_t) I, I\rangle \, dt,$$

where $z_t = z_0 + tI$, $z_1 = z$, I varies in a small ball $B_R(0) = \{|I| < R\}$ in $\mathbf{R}^n$, and $\nabla_z^2 H^0$ stands for the Hessian matrix of H^0. We put $\omega = \nabla H^0(z_0)$. Then $z_0 = \nabla g^0(\omega)$, g^0 being the Legendre transform of H^0, and we write

$$H^0(z) = e(\omega) + \langle \omega, I\rangle + P_{H^0}(I; \omega),$$
$$H^1(\theta, z) = H^1(\theta, \nabla g(\omega) + I) = P_{H^1}(\theta, I; \omega),$$

where $e(\omega) = H^0(\nabla g(\omega))$, while P_{H^0} stands for the quadratic term in I in the expression of H^0. We set $P = P_{H^0} + P_{H^1}$ and consider the family of Hamiltonians

$$H(\theta, I; \omega) := e(\omega) + \langle \omega, I\rangle + P(\theta, I; \omega) \tag{2.1}$$

in $\mathbf{T}^n \times B_R(0)$ depending on the frequency $\omega \in \Omega$. From now on, to simplify the notations, we replace $C\epsilon_H$, CA_0, CL_1 and CL_2 by ϵ_H, A_0, L_1 and L_2, respectively, whenever $C \geq 1$ depends only on L_0, ρ, τ and n. Then using (1.1), (1.2), and Proposition A.3 we obtain

$$|\partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma P(\theta, I; \omega)| \leq \left(A_0 R^2 + \kappa^2 \epsilon_H\right) L_1^{|\alpha|} L_2^{|\beta|+|\gamma|} (\alpha!\beta!\gamma!)^\rho,$$

for any α, β and γ, and uniformly with respect to $(\theta, I; \omega) \in \mathbf{T}^n \times B_R(0) \times \Omega$. Hence, we can suppose that $P \in \mathcal{G}^\rho_{L_1, L_2, L_2}(\mathbf{T}^n \times B \times \Omega)$, $B = B_R(0)$, with norm

$$\|P\| = \sup \left(|\partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma P(\theta, I; \omega)| L_1^{-|\alpha|} L_2^{-|\beta|-|\gamma|} (\alpha!\beta!\gamma!)^{-\rho}\right) \leq A_0 R^2 + \kappa^2 \epsilon_H, \tag{2.2}$$

where the sup is taken over all multi-indices α, β, γ and for all $(\theta, I; \omega) \in \mathbf{T}^n \times B \times \Omega$. Fix $0 < \varsigma \leq 1$. We can now formulate our main result in this section.

Theorem 2.1 *Suppose that H is given by (2.1) where $P \in \mathcal{G}^{\rho}_{L_1,L_2,L_2}(\mathbf{T}^n \times \overline{B}_R(0) \times \overline{\Omega})$. Fix $\kappa > 0$ and $r > 0$ such that $\kappa, r < L_2^{-1-\varsigma}$ and $r \le R$. Then there is $N = N(n, \rho, \tau) > 0$ and $\epsilon > 0$ independent of κ, L_1, L_2, r, R, and of $\Omega \subset \Omega_0$, such that if*

$$\|P\| \le \ \epsilon\kappa r L_1^{-N} \tag{2.3}$$

then there exist maps $\phi \in \mathcal{G}^{\rho'}(\Omega, \Omega)$ and $\Phi = (U, V) \in \mathcal{G}^{\rho,\rho'}(\mathbf{T}^n \times \Omega, \mathbf{T}^n \times B_R(0))$, $\rho' = \rho(\tau+1)+1$, satisfying

(i) For each $\omega \in \Omega_\kappa$, the map $\Phi_\omega := \Phi(\cdot, \omega) : \mathbf{T}^n \to \mathbf{T}^n \times B_R(0)$ is an $\mathcal{G}^\rho$ embedding and $\Lambda_\omega := \Phi_\omega(\mathbf{T}^n)$ is an embedded Lagrangian torus invariant with respect to the Hamiltonian flow of $H_{\phi(\omega)}(\varphi, I) := H(\varphi, I; \phi(\omega))$. Moreover, $X_{H_{\phi(\omega)}} \circ \Phi_\omega = D\Phi_\omega \cdot \mathcal{L}_\omega$ on $\mathbf{T}^n$.

(ii) There exist $A, C > 0$, independent of κ, L_1, L_2, r, and of $\Omega \subset \Omega_0$, such that

$$\left|\partial_\theta^\alpha \partial_\omega^\beta (U(\theta;\omega) - \theta)\right| + r^{-1}\left|\partial_\theta^\alpha \partial_\omega^\beta V(\theta;\omega)\right| + \kappa^{-1}\left|\partial_\omega^\beta (\phi(\omega) - \omega)\right|$$

$$\le \ A\, C_1^{|\alpha|}\ (C_2\kappa^{-1})^{|\beta|}\ \alpha!^{\,\rho} \beta!^{\,\rho'}\ \frac{\|P\| L_1^N}{\kappa r}$$

uniformly in $(\theta, \omega) \in \mathbf{T}^n \times \Omega$ and for any α and β, where $C_1 = CL_1$ and $C_2 = CL_1^{\tau+1}$.

Remark. Note that the constant $e(\omega)$ in (2.1) plays no role in Theorem 2.1 and from now on we suppose $e(\omega) \equiv 0$.

Theorem 2.1 will be proved in the next section. Theorem 1.1 and Corollary 1.2 follow from Theorem 2.1 and they will be proved in Sect. 4.

3 *Proof of Theorem 2.1.*

We divide the proof of Theorem 2.1 in several steps. First, using Theorem 3.7, we extend P to a Gevrey function $\widetilde{P}$ of the class $\mathcal{G}^{\rho}_{\widetilde{L}_1,\widetilde{L}_2}(\mathbf{T}^n \times \mathbf{R}^{2n})$ such that $\|\widetilde{P}\| \le A\|P\|$, $\widetilde{L}_1 = CL_1$, and $\widetilde{L}_2 = CL_2$, where the constants $A, C > 0$ are independent of P, R and Ω. To simplify the notations we drop $\sim$. Multiplying P with a suitable cut-off function we assume that the support of P with respect to (I, ω) is contained in $B_1(0) \times B_{\bar R}(0)$, $\bar R \gg 1$.

3.1 *Approximation Lemma for Gevrey functions*

We fix $0 < \varsigma \le 1$ and choose three strictly decreasing sequences of positive numbers $\{u_j\}_{j=0}^\infty$, $\{v_j\}_{j=0}^\infty$ and $\{w_j\}_{j=0}^\infty$ tending to 0 and such that

$$\forall j \in \mathbf{N}\ :\quad v_j L_2,\, w_j L_2 \le u_j L_1 \le 1\,,\quad v_0,\, w_0 \le L_2^{-1-\varsigma}\,. \tag{3.1}$$

Consider the complex sets $\mathcal{U}_j^m$, $m = 1, 2$, in $\mathbf{C}^n/2\pi\mathbf{Z}^n \times \mathbf{C}^n \times \mathbf{C}^n$ consisting of all (θ, I, ω) with real parts $\mathrm{Re}\,\theta \in \mathbf{T}^n$, $\mathrm{Re}\, I \in B_2(0)$ and $\mathrm{Re}\,\omega \in B_{\bar R+1}(0)$, and such that $|\mathrm{Im}\,\theta_k| \le m u_j$, $|\mathrm{Im}\, I_k| \le m v_j$, $|\mathrm{Im}\,\omega_k| \le m w_j$, for each $1 \le k \le n$. Set $\mathcal{U}_j = \mathcal{U}_j^1$ and denote by $A(\mathcal{U}_j)$ the set of all real-analytic bounded functions in $\mathcal{U}_j$ equipped with the sup-norm $|\cdot|_{\mathcal{U}_j}$.

Proposition 3.1 (Approximation Lemma) *Let $P \in \mathcal{G}^{\rho}_{L_1,L_2}(\mathbf{T}^n \times \mathbf{R}^{2n})$. Suppose that the support of P with respect to (I,ω) is in $B_1(0) \times B_{\bar{R}}(0)$, and assume (3.1). Then there is a sequence $P_j \in A(\mathcal{U}_j)$, $j \geq 0$, such that*

$$\begin{aligned} |P_{j+1} - P_j|_{\mathcal{U}_{j+1}} &\leq C_0\, L_1^n \exp\left(-\tfrac{3}{4}(\rho-1)(2L_1 u_j)^{-\frac{1}{\rho-1}}\right) \|P\|, \\ |P_0|_{\mathcal{U}_0} &\leq C_0 \left(1 + L_1^n \exp\left(-\tfrac{3}{4}(\rho-1)(2L_1 u_0)^{-\frac{1}{\rho-1}}\right)\right) \|P\|. \end{aligned}$$

where $C_0 = \widetilde{C}_0(n,\rho,\varsigma)(\bar{R}^n + 1)$. Moreover,

$$\sup |\partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma (P - P_j)(\theta, I, \omega)| \;\leq\; C_0\, L_1^n\, L_2 \exp\left(-\frac{3}{4}(\rho-1)(2L_1 u_j)^{-\frac{1}{\rho-1}}\right) \|P\|,$$

in $\mathbf{T}^n \times B_1(0) \times B_{\bar{R}}(0)$ for $|\alpha| + |\beta| + |\gamma| \leq 1$.

Remark. Instead of 3/4 we can take above any positive number less than 1 in order to absorb certain polynomials of $(L_1 u_j)^{-1}$. Similar estimates can be obtained without the inequalities $v_0, w_0 \leq L_2^{-1-\varsigma}$. In this case $\widetilde{C}_0 = \widetilde{C}_0(n,\rho)$ but the right hand side of the estimates above should be multiplied by L_2^{2n}. Using the 'standard' proof of the Approximation Lemma [12] one obtains for any $\delta > 0$ an approximation modulo $C(\rho,\delta) \exp\left(-c(\rho)(L_1 u_j)^{-\frac{1}{\rho}+\delta}\right) \|P\|$, $C, c > 0$.

Proof. We divide the proof into two parts.

1. Almost analytic extension of P. There is a constant $C(\rho) \geq 1$, depending only on ρ, such that

$$t \in (0,2]\,, \; m \in \mathbf{N}\,, \quad 1 \leq m \leq t^{-\frac{1}{\rho-1}} + 1\,, \tag{3.2}$$

implies

$$t^m\, m!^{\,\rho-1} \;\leq\; C(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m}. \tag{3.3}$$

Indeed, by Stirling's formula, we get

$$\begin{aligned} t^m\, m!^{\,\rho-1} &\leq C_1(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m} \exp\{m[(\rho-1)\ln m + \ln t]\} \\ &= C_1(\rho) m^{(\rho-1)/2} e^{-(\rho-1)m} \exp\left\{(\rho-1) m \ln\left[m\, t^{\frac{1}{\rho-1}}\right]\right\}. \end{aligned}$$

Moreover,

$$m \ln\left[m\, t^{\frac{1}{\rho-1}}\right] \leq m \ln\left[1 + t^{\frac{1}{\rho-1}}\right] \leq m\, t^{\frac{1}{\rho-1}} \leq 1 + 2^{\frac{1}{\rho-1}}\,,$$

which proves (3.3).

We define an almost analytic extensions F_j of P in $\mathcal{U}_j^2$ as follows

$$F_j(\theta + i\widetilde{\theta}, I + i\widetilde{I}, \omega + i\widetilde{\omega}) \;=\; \sum_{(\alpha,\beta,\gamma)\in\mathcal{M}_j} \partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma P(\theta, I, \omega) \frac{(i\widetilde{\theta})^\alpha (i\widetilde{I})^\beta (i\widetilde{\omega})^\gamma}{\alpha!\beta!\gamma!}\,. \tag{3.4}$$

The index set $\mathcal{M}_j$ consists of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ such that $\alpha_k \leq N_1$, $\beta_k \leq N_2$ and $\gamma_k \leq N_3$, $k = 1, \ldots, n$, where

$$N_1 = \left[(2L_1 u_j)^{-\frac{1}{\rho-1}}\right] + 1\,,\; N_2 = \left[(2L_2 v_j)^{-\frac{1}{\rho-1}}\right] + 1\,,\; N_3 = \left[(2L_2 w_j)^{-\frac{1}{\rho-1}}\right] + 1\,, \tag{3.5}$$

and $[t]$ stands for the integer part of t. We have

$$|F_j|_{\mathcal{U}_j^2} \;\le\; \|P\| \sum_{(\alpha,\beta,\gamma)\in\mathcal{M}_j} (2L_1u_j)^{|\alpha|}(2L_2v_j)^{|\beta|}(2L_2w_j)^{|\gamma|}(\alpha!\beta!\gamma!)^{\rho-1}\ .$$

For $\alpha_k, \beta_k, \gamma_k \neq 0$ we estimate each term

$$(2L_1u_j)^{\alpha_k}\alpha_k!^{\,\rho-1}\ ,\ (2L_2v_j)^{\beta_k}\beta_k!^{\,\rho-1}\ ,\ (2L_2w_j)^{\gamma_k}\gamma_k!^{\,\rho-1}\ ,\ k=1,\dots,n\,,$$

by $C(\rho)m^{(\rho-1)/2}e^{-(\rho-1)m}$, where $m\ge 1$ stands for α_k, β_k, and γ_k respectively. To this end, we put $t = 2L_1u_j$, $2L_2v_j$, $2L_2w_j$, respectively, and we get $t\in(0,2]$ in view of (3.1). Now (3.2) holds because of (3.5). Then using (3.3) we obtain

$$|F_j|_{\mathcal{U}_j^2} \;\le\; \|P\|\left(1+C(\rho)\sum_{m=1}^{\infty} m^{(1-\rho)/2}e^{-(\rho-1)m}\right)^{3n} \;=\; C_1\|P\|.$$

Set $z_k = \theta_k + i\widetilde{\theta}_k$. Then applying $\bar\partial_{z_k} = (\partial/\partial\theta_k + i\partial/\partial\widetilde{\theta}_k)/2$ to F_j we obtain

$$2\bar\partial_{z_k}F_j(\theta+i\widetilde{\theta}, I+i\widetilde{I}, \omega+i\widetilde{\omega}) \;=\; \sum_{\substack{(\alpha,\beta,\gamma)\in\mathcal{M}_j\\ \alpha_k=N_1}} \partial_\theta^\alpha\partial_I^\beta\partial_\omega^\gamma\partial_{\theta_k}P(\theta,I,\omega)\frac{(i\widetilde{\theta})^\alpha(i\widetilde{I})^\beta(i\widetilde{\omega})^\gamma}{\alpha!\beta!\gamma!}\ . \tag{3.6}$$

We estimate each term in the sum by

$$L_1(2L_1u_j)^{|\alpha|}(2L_2v_j)^{|\beta|}(2L_2w_j)^{|\gamma|}(\alpha!\beta!\gamma!)^{\rho-1}(\alpha_k+1)^\rho\|P\|$$

in $\mathcal{U}_j^2$, where $\alpha_k = N_1$. Since

$$(2L_1u_j)^{-\frac{1}{\rho-1}} \le \alpha_k = N_1 \le (2L_1u_j)^{-\frac{1}{\rho-1}}+1,$$

we obtain from (3.3) (with $t = 2L_1u_j$ and $m = N_1$)

$$(2L_1u_j)^{\alpha_k}\,\alpha_k!^{\,\rho-1}(\alpha_k+1)^\rho \le C'(L_1u_j)^{-\frac{\rho}{\rho-1}-\frac12}\exp\left(-(\rho-1)(2L_1u_j)^{-\frac{1}{\rho-1}}\right).$$

This implies as above

$$|\bar\partial_{z_k}F_j|_{\mathcal{U}_j^2} \;\le\; C''L_1(L_1u_j)^{-\frac{\rho}{\rho-1}-\frac12}\exp\left(-(\rho-1)(2L_1u_j)^{-\frac{1}{\rho-1}}\right)\ \|P\|$$

$$|\bar\partial_{z_k}F_j|_{\mathcal{U}_j^2} \;\le\; C\,L_1\ \exp\left(-\tfrac34\,(\rho-1)(2L_1u_j)^{-\frac{1}{\rho-1}}\right)\ \|P\|\,,$$

where $C = C(\rho,n)>0$. In the same way, differentiating (3.6), we get with $\beta+\gamma=1$

$$|\partial_{\theta_p}^\beta\partial_{\widetilde{\theta}_p}^\gamma\,\bar\partial_{z_k}F_j|_{\mathcal{U}_j^2}\ ,\ |\partial_{I_p}^\beta\partial_{\widetilde{I}_p}^\gamma\,\bar\partial_{z_k}F_j|_{\mathcal{U}_j^2}\ ,\ |\partial_{\omega_p}^\beta\partial_{\widetilde{\omega}_p}^\gamma\,\bar\partial_{z_k}F_j|_{\mathcal{U}_j^2}$$
$$\le CL_1L_2\exp\left(-\tfrac34\,(\rho-1)(2L_1u_j)^{-\frac{1}{\rho-1}}\right)\ \|P\|\,,$$

where $C = C(\rho,n)>0$ (we recall that $L_2\ge L_1\ge 1$). Using (3.1), we obtain the same estimates for $\bar\partial_{I_k}F_j$, $\bar\partial_{\omega_k}F_j$, and for their derivatives of order one in $\mathcal{U}_j^2$. Indeed, putting $z_k = I_k + i\widetilde{I}_k$ we obtain

$$|\bar\partial_{z_k}F_j|_{\mathcal{U}_j^2} \le C'L_2(L_2v_j)^{-\frac{\rho}{\rho-1}-\frac12}\exp\left(-(\rho-1)(2L_2v_j)^{-\frac{1}{\rho-1}}\right)\ \|P\|$$

$$\le C\exp\left(-\tfrac34(\rho-1)(2L_2v_j)^{-\frac{1}{\rho-1}}\right)\|P\| \le C\exp\left(-\tfrac34(\rho-1)(2L_1u_j)^{-\frac{1}{\rho-1}}\right)\|P\|\,,$$

where $C = C(\rho, n, \varsigma) > 0$, since $L_2 \le (L_2 v_j)^{-1/\varsigma}$ by (3.1). We generalize these estimates as follows. Set $z = (\theta, I, \omega) \in \mathbf{C}^n/2\pi\mathbf{Z}^n \times \mathbf{C}^{2n}$, denote by x_k and y_k, respectively, the real and the imaginary part of z_k, $1 \le k \le 3n$, and put $\bar\partial_{z_k} = (\partial_{x_k} + i\partial_{y_k})/2$. Then using (3.1), (3.3) and (3.5), we obtain for any $\delta = (\delta_1, \dots, \delta_{3n}) \in \mathbf{N}^{3n}$ with $0 \le \delta_k \le 1$ and $|\delta| \ge 1$ the estimate

$$|\,\bar\partial_z^\delta F_j\,|_{\mathcal{U}_j^2} \;\le\; C\, L_1^n \exp\left(-\frac{3}{4}(\rho-1)(2L_1 u_j)^{-\frac{1}{\rho-1}}\right) \|P\|,$$

where $C = C(n, \rho, \varsigma) > 0$. To this end, differentiating (3.4), we obtain an expression similar to (3.6), where for each k such that $\delta_k = 1$ we have $\alpha_k = N_1$ if $1 \le k \le n$, $\alpha_k = N_2$ if $n+1 \le k \le 2n$, and $\alpha_k = N_3$ if $2n+1 \le k \le 3n$, and then we proceed as above. More generally, for any $\delta = (\delta_1, \dots, \delta_{3n}) \in \mathbf{N}^{3n}$ with $0 \le \delta_k \le 1$ and $|\delta| \ge 1$, and any $\beta, \gamma \in \mathbf{N}^{3n}$ with $0 \le |\beta| + |\gamma| \le 1$, we obtain as above the estimate

$$|\partial_x^\beta \partial_y^\gamma \bar\partial_z^\delta F_j|_{\mathcal{U}_j^2} \;\le\; C\, L_1^n\, L_2^{|\beta|+|\gamma|} \exp\left(-\frac{3}{4}(\rho-1)(2L_1 u_j)^{-\frac{1}{\rho-1}}\right) \|P\|, \tag{3.7}$$

where $C = C(n, \rho, \varsigma) > 0$. Obviously, the same estimate holds for $\bar\partial_z^\beta \partial_z^\gamma \bar\partial_z^\delta F_j$ if $0 \le |\beta| + |\gamma| \le 1$.

2. Construction of P_j. We are going to approximate F_j by analytic in $\mathcal{U}_j^2$ functions using Green's formula

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(\eta)}{\eta - \zeta}\, d\eta + \frac{1}{2\pi i}\iint_D \frac{\bar\partial_\eta f(\eta)}{\eta - \zeta}\, d\eta \wedge d\bar\eta = \begin{cases} f(\zeta) & \text{if } \zeta \in D \\ 0 & \text{if } \zeta \notin \bar D, \end{cases} \tag{3.8}$$

where $D \subset \mathbf{C}$ is a bounded domain with a piecewise smooth boundary ∂D which is positively oriented with respect to D, $\bar D = D \cup \partial D$, and $f \in C^1(\bar D)$.

We denote by $D_k \subset \mathbf{C}$ the open rectangle $\{|x_k| < a_k,\ |y_k| < b_k\}$, where $a_k = \pi$ and $b_k = 2u_j$ for $1 \le k \le n$; $a_k = 2$ and $b_k = 2v_j$ for $n+1 \le k \le 2n$, and $a_k = \bar R + 1$ and $b_k = 2w_j$ for $2n+1 \le k \le 3n$. We denote also by ∂D_k the boundary of D_k which is positively oriented with respect to D_k and by Γ the union of the oriented segments $[-\pi - 2iu_j, \pi - 2iu_j] \cup [\pi + 2iu_j, -\pi + 2iu_j]$. Note that D_k and Γ depend on j as well but we omit it. Given $\eta \in \mathbf{C}$, we consider the 2π-periodic meromorphic function

$$\zeta \mapsto K(\eta, \zeta) = \frac{1}{\eta - \zeta} + K_1(\eta, \zeta),\ K_1(\eta, \zeta) = \lim_{N \to +\infty} \sum_{k=1}^N \left(\frac{1}{\eta - \zeta + 2\pi k} + \frac{1}{\eta - \zeta - 2\pi k}\right).$$

Consider the function

$$F_{j,1}(z) := \frac{1}{2\pi i}\int_\Gamma F_j(\eta_1, z_2, \dots, z_{3n}) K(\eta_1, z_1)\, d\eta_1\ , \quad z \in \mathcal{U}_j^2.$$

It is analytic and 2π-periodic with respect to z_1 in the strip $\{|\mathrm{Im}\, z_1| < 2u_j\}$. Moreover, for $z_1 \in D_1$, we have

$$F_{j,1}(z) = \frac{1}{2\pi i}\int_{\partial D_1} F_j(\eta_1, z_2, \dots, z_{3n}) K(\eta_1, z_1)\, d\eta_1$$

since the function under the integral is 2π-periodic with respect to η_1, and using (3.8) we obtain

$$F_{j,1}(z) = F_j(z) - \frac{1}{2\pi i}\int_{D_1} \bar\partial_{\eta_1} F_j(\eta_1, z_2, \dots, z_{3n}) K(\eta_1, z_1)\, d\eta_1 \wedge d\bar\eta_1.$$

By continuity last formula remains true for $\operatorname{Re} z_1 = \pm\pi$. Set $F_{j,0}(z) := F_j(z)$ and $\mathcal{U}^2_{j,1} := \mathcal{U}^2_j \cap \{|\operatorname{Im} z_1| \le u_j\}$. We claim that for any multi-index $\alpha = (0, \alpha_2, \dots, \alpha_{3n}) \in \mathbf{N}^{3n}$ with $0 \le \alpha_m \le 1$, $1 \le m \le 3n$, any index k, and $\beta, \gamma \in \mathbf{N}$ such that $0 \le \beta + \gamma \le 1$, we have

$$\left|\partial^\beta_{z_k} \bar\partial^\gamma_{z_k} \bar\partial^\alpha_z (F_{j,1} - F_{j,0})\right|_{\mathcal{U}^2_{j,1}} \le C\, L_1^n\, L_2^{\beta+\gamma} \exp\left(-\frac{3}{4}(\rho-1)(2L_1 u_j)^{-\frac{1}{\rho-1}}\right) \|P\|, \tag{3.9}$$

where $C = C(n, \rho, \varsigma) > 0$. For $k \neq 1$ it directly follows from (3.7) differentiating under the integral. To prove it for $k = 1$, we use the same argument for

$$\frac{1}{2\pi i} \int_{D_1} \bar\partial_{\eta_1} F_j(\eta_1, z_2, \dots, z_{3n}) K_1(\eta_1, z_1)\, d\eta_1 \wedge d\bar\eta_1.$$

On the other hand,

$$\begin{aligned} &\frac{1}{2\pi i} \int_{D_1} \frac{\bar\partial_{\eta_1} F_j(\eta_1, z_2, \dots, z_{3n})}{\eta_1 - z_1}\, d\eta_1 \wedge d\bar\eta_1 \\ &= -\bar z_1 \bar\partial_{z_1} F_j(z) + \frac{1}{2\pi i} \int_{D_1} \frac{\bar\partial_{\eta_1} F_j(\eta_1, z_2, \dots, z_{3n}) - \bar\partial_{z_1} F_j(z)}{\eta_1 - z_1}\, d\eta_1 \wedge d\bar\eta_1 \end{aligned}$$

for $z_1 \in D_1$, which follows from (3.8) applied to $f(z_1) = \bar z_1$. Differentiating the last equality and using (3.7) we get the estimate. Moreover, if $|\alpha| \ge 1$, then (3.7) and (3.9) imply

$$\left|\partial^\beta_{z_k} \bar\partial^\gamma_{z_k} \bar\partial^\alpha_z F_{j,1}\right|_{\mathcal{U}^2_{j,1}} \le C\, L_1^n\, L_2^{\beta+\gamma} \exp\left(-\frac{3}{4}(\rho-1)(2L_1 u_j)^{-\frac{1}{\rho-1}}\right) \|P\|. \tag{3.10}$$

We define by recurrence $F_{j,m}(z)$, $2 \le m \le n$, and we prove that it satisfies (3.9) in $\mathcal{U}^2_{j,m} := \mathcal{U}^2_{j,m-1} \cap \{|\operatorname{Im} z_m| \le u_j\}$ for $\alpha = (0, \dots, 0, \alpha_{m+1}, \dots, \alpha_{3n})$. Moreover, $F_{j,m}(z)$ satisfies (3.10) for $|\alpha| \ge 1$.

For $n < m \le 3n$ we define

$$F_{j,m}(z) = \frac{1}{2\pi i} \int_{\partial D_m} \frac{F_{j,m-1}(z_1, \dots, z_{m-1}, \eta_m, z_{m+1}, \dots, z_{3n})}{\eta_m - z_m}\, d\eta_m\ , \quad z \in \mathcal{U}^2_j,$$

and set $\mathcal{U}^2_{j,m} := \mathcal{U}^2_{j,m-1} \cap \{|\operatorname{Im} z_m| \le p_m\}$, where $p_m = v_j$ for $n+1 \le m \le 2n$ and $p_m = w_j$ for $2n+1 \le m \le 3n$. By recurrence with respect to m, we obtain (3.9) for $F_{j,m}$ in $\mathcal{U}^2_{j,m}$ for any $n < m \le 3n$, $\alpha = (0, \dots, 0, \alpha_{m+1}, \dots, \alpha_{3n})$, $0 \le \alpha_q \le 1$ ($\alpha = 0$ if $m = 3n$), for any index k and β, γ such that $0 \le \beta + \gamma \le 1$. Moreover, $F_{j,m}(z)$ satisfies (3.10) for $|\alpha| \ge 1$ and $m < 3n$. For $2n < m \le 3n$ the constant C should be replaced by $C'(1 + \bar R)^{m-2n}$, where $C' = C'(\rho, n, \varsigma)$. The factor $1 + \bar R$ comes from the measure of D_m, $2n < m \le 3n$.

Set $P_j = F_{j,3n}$. Then for any index k and $\ell = 0, 1$, we obtain

$$|\partial^\ell_{x_k} (P_j - F_j)|_{\mathcal{U}_j} \le C\, L_1^n\, L_2^\ell \exp\left(-\frac{3}{4}(\rho-1)(2L_1 u_j)^{-\frac{1}{\rho-1}}\right) \|P\|.$$

In particular,

$$\begin{aligned} |P_{j+1} - P_j|_{\mathcal{U}_{j+1}} &\le |P_{j+1} - F_{j+1}| + |P_j - F_j| + |F_{j+1} - F_j| \\ &\le C\, L_1^n \exp\left(-\tfrac{3}{4}(\rho-1)(2L_1 u_j)^{-\frac{1}{\rho-1}}\right) \|P\|. \end{aligned}$$

Moreover, for any index k and $\ell = 0, 1$,

$$|\partial^{\ell}_{x_k}(P_j(x) - P(x))| \;\leq\; C\, L_1^n\, L_2^{\ell} \exp\left(-\frac{3}{4}(\rho-1)(2L_1u_j)^{-\frac{1}{\rho-1}}\right) \|P\|$$

in $\mathcal{U}_j \cap \{\operatorname{Im} z = 0\}$, since $F_j(x) = P(x)$ for x real. Finally,

$$|P_0|_{\mathcal{U}_0} \leq |F_0| + |P_0 - F_0| \leq \; C\left(1 + L_1^n \exp\left(-\frac{3}{4}(\rho-1)(2L_1u_0)^{-\frac{1}{\rho-1}}\right)\right) \|P\|.$$

This completes the proof of the proposition. $\square$

3.2 *The KAM step*

Introduce the complex domains

$$D_{s,r} = \{\theta \in \mathbf{C}^n/2\pi\mathbf{Z}^n :\; |\operatorname{Im}\theta| < s\} \times \{I \in \mathbf{C}^n :\; |I| < r\},$$

$$O_h = \{\omega \in \mathbf{C}^n :\; |\omega - \Omega_\kappa| < h\}.$$

The sup-norm of functions in $\mathcal{V} := D_{s,r} \times O_h$ will be denoted by $|\cdot|_{s,r,h}$. Fix $0 < \upsilon < 1/6$ and set $\widetilde{\upsilon} = 1/2 - 3\upsilon$ (we shall choose later $\upsilon = 1/54$ and $\widetilde{\upsilon} = 4/9$). Fix $0 < s, r < 1,\; 0 < \eta < 1/8,\; 0 < \sigma < s/5,\; K \geq 1$. Consider the real valued Hamiltonian $H(\theta, I; \omega) = N(I;\omega) + H_1(\theta, I; \omega)$, $N(I;\omega) = e(\omega) + \langle\omega, I\rangle$. We shall denote by 'Const.' a positive constant depending only on n and τ and by 'const.' if it is ≤ 1. We recall from Pöschel [9] the following

Proposition 3.2 *Let H be real analytic in $\mathcal{V}$. Suppose that $|H - N|_{s,r,h} \leq \varepsilon$ with*

(a) $\varepsilon \leq \text{const.}\, \kappa\eta r\sigma^{\tau+1}$,

(b) $\varepsilon \leq \text{const.}\, \upsilon h r$,

(c) $h \leq \dfrac{\kappa}{2K^{\tau+1}}$

Then there exists a real analytic transformation

$$\mathcal{F} = (\Phi, \phi)\;, \quad \Phi : D_{s-5\sigma,\eta r} \times O_h \longrightarrow D_{s,r} \times O_h\;,\; \phi : O_{\widetilde{\upsilon}h} \longrightarrow O_h\;,$$

of the form $\Phi(\theta, I; \omega) = (U(\theta;\omega), V(\theta, I;\omega))$, with V affine linear with respect to I, where the transformation $\Phi(\cdot;\omega)$ is canonical for each ω, and such that $H \circ \mathcal{F} = N_+ + P_+$ with $N_+(I;\omega) = e_+(\omega) + \langle\omega, I\rangle$, and

$$|P_+|_{s-5\sigma,\eta r,\widetilde{\upsilon}h} \leq \text{Const.}\left(\frac{\varepsilon^2}{\kappa r\sigma^{\tau+1}} + (\eta^2 + K^n e^{-K\sigma})\varepsilon\right). \tag{3.11}$$

Moreover,

$$|W(\Phi - id)|,\; |W(D\Phi - \text{Id})W^{-1}| \leq \text{Const.}\,\frac{\varepsilon}{\kappa r\sigma^{\tau+1}}\,,$$

$$|\phi - id|,\; \upsilon h|D\phi - \text{Id}| \leq \text{Const.}\,\frac{\varepsilon}{r}\,,$$

uniformly on $D_{s-5\sigma,\eta r} \times O_h$ and $O_{\widetilde{\upsilon}h}$, respectively, where $W = \text{diag}\,(\sigma^{-1}\text{Id}\,, r^{-1}\text{Id}\,)$.

Remark 3.3 *Set $\overline{W} = \mathrm{diag}\,(\sigma^{-1}\mathrm{Id}\,, r^{-1}\mathrm{Id}\,, h^{-1}\mathrm{Id})$ and suppose that $h \le \kappa\sigma^{\tau+1}$. Since $1 - \widetilde{\upsilon} \ge 1/3$, using the Cauchy estimate with respect to ω, we obtain*

$$|\overline{W}(\mathcal{F} - id)|,\ |\overline{W}(D\mathcal{F} - \mathrm{Id})\overline{W}^{-1}| \le \frac{C\varepsilon}{rh}\,,\ C = C(n,\tau) > 0\,,$$

uniformly on $D_{s-5\sigma,\eta r} \times O_{\widetilde{\upsilon}h}$, where $D\mathcal{F}$ stands for the Jacobian of $\mathcal{F}$.

The proof of Proposition 3.2 is given in [9]. The only difference between the statement of Proposition 1.3 and that of the KAM step in [9] appears in the transformation of the frequencies ($\widetilde{\upsilon} = 1/4$ in [9]). To prove the proposition with $\widetilde{\upsilon}$ as above we use the following analog of Lemma A.3 [9].

Lemma 3.4 *Suppose $f : O_h \to \mathbf{C}^n$ is real analytic with bounded $|f|_h$. Let $0 < \upsilon < 1/6$ and $\widetilde{\upsilon} = 1/2 - 3\upsilon$. If $|f - id|_h \le \upsilon h$, then f has a real analytic inverse $f : O_{\widetilde{\upsilon}h} \to O_{2(\upsilon+\widetilde{\upsilon})h}$ and*

$$|\phi - id|_{\widetilde{\upsilon}h}\ ,\ 3\upsilon h|D\phi - id|_{\widetilde{\upsilon}h} \le |f - id|_h\,.$$

A sketch of proof of the Lemma is given in the Appendix. We are going to prepare the next iteration. We choose a 'weighted error' $0 < E < 1$, fix $0 < \widehat{\varepsilon} \le 1$, and set

$$\eta = E^{1/2},\ \varepsilon = \widehat{\varepsilon}\kappa E r\sigma^{\tau+1},\ 0 < E < 1/64.$$

We define K and h by

$$K^n e^{-K\sigma} = E,\ h = \frac{\kappa}{2K^{\tau+1}}.$$

Setting $x = K\sigma$ we get the equation $x^n e^{-x} = E\sigma^n$, which has an unique solution with respect to $x \in [1, +\infty)$, since $0 < E < 1/64 < 1/e$. Then $K = x\sigma^{-1} > 1$. We set

$$r_+ = \eta r,\ s_+ = s - 5\sigma,\ \sigma_+ = \delta\sigma,$$

where $0 < \delta < 1$. Later we shall choose $\delta = \delta(\rho)$ as a function of ρ only. Now the KAM step gives the estimate

$$|P_+|_{s_+,r_+,\widetilde{\upsilon}h} < \mathrm{Const.}\,\widehat{\varepsilon}\kappa r\sigma^{\tau+1}\left(E^2 + (\eta^2 + K^n e^{-K\sigma})E\right) = \mathrm{Const.}\,\widehat{\varepsilon}\kappa r\sigma^{\tau+1}E^2$$

$$= \mathrm{Const.}\,\delta(\rho)^{-\tau-1}\widehat{\varepsilon}\kappa r_+\sigma_+^{\tau+1}E^{3/2}.$$

Hence there is a constant $c_1 > 1$ depending only on n, ρ and τ such that

$$|P_+|_{s_+,r_+,\widetilde{\upsilon}h} \le \frac{1}{2}\widehat{\varepsilon}c_1^{1/2}\kappa r_+\sigma_+^{\tau+1}E^{3/2}.$$

We fix the weighted error for the iteration by $E_+ = c_1^{1/2}E^{3/2}$, set $\varepsilon_+ = \widehat{\varepsilon}\kappa r_+\sigma_+^{\tau+1}E_+$, and then define $\eta_+,\ x_+,\ K_+$, and h_+ as above. Notice that, $c_1E_+ = (c_1E)^{3/2}$. We require also $c_1E < 1$ which leads to an exponentially converging scheme. Suppose that

$$h_+ \ \le\ \widetilde{\upsilon}\,h. \tag{3.12}$$

Then we obtain

$$|P_+|_{s_+,r_+,h_+}\ \le\ \frac{1}{2}\varepsilon_+. \tag{3.13}$$

3.3 *Setting the parameters and iteration*

As in [9] we are going to iterate the KAM step infinitely many times choosing appropriately the parameters $0 < s, r, \sigma, h, \eta < 1$ and so on. Our goal is to get a convergent scheme in the Gevrey spaces $\mathcal{G}^{\rho,\rho(\tau+1)+1}$. We are going to define suitable strictly decreasing sequences of positive numbers $\{s_j\}_{j=0}^{\infty}$, $\{r_j\}_{j=0}^{\infty}$ and $\{h_j\}_{j=0}^{\infty}$, tending to 0, and denote

$$D_j = D_{s_j, r_j}\ ,\ O_j = O_{h_j}\ ,\ \mathcal{V}_j = D_j \times O_j .$$

Fix $\delta \in (0,1)$ (δ will depend only on ρ) and set

$$s_j = s_0 \delta^j\ ,\ \sigma_j = \sigma_0 \delta^j\ ,\ s_0(1-\delta) = 5\sigma_0\, .$$

Obviously, $s_{j+1} = s_j - 5\sigma_j$ and $\sigma_j = 5^{-1}(1-\delta)s_j$ for $j \geq 0$. We set

$$u_j = 4s_j = 4s_0\delta^j\ ,\ v_j = 4r_0\delta^j\ ,\ w_j = 4h_0\delta^j,$$

and denote by $\mathcal{U}_j$ the corresponding complex sets defined in Sect. 3.1. We assume for the moment that these sequences verify (3.1). Then applying Proposition 3.1 we obtain

$$\begin{array}{rcl} |P_0|_{\mathcal{U}_0} & \leq & C_0\, L_1^n\, \|P\| \\ |P_j - P_{j-1}|_{\mathcal{U}_j} & \leq & C_0\ L_1^n\, \|P\|\, e^{-\widetilde{B}_0\, s_j{}^{-\frac{1}{\rho-1}}} \ = \ C_0\ L_1^n\, \|P\|\, e^{-B_0\, \sigma_j{}^{-\frac{1}{\rho-1}}}\ \ ,\ j \geq 1\, , \end{array} \tag{3.14}$$

where the positive constants $\widetilde{B}_0 L_1^{\frac{1}{\rho-1}}$ and $B_0 L_1^{\frac{1}{\rho-1}}$ depend only on ρ and δ. Given N and $a > 0$ we set

$$\widehat{\varepsilon} := \|P\| L_1^{N-2} (a\kappa r)^{-1} \leq 1\, , \tag{3.15}$$

and we introduce

$$\widetilde{\varepsilon}_j = \widehat{\varepsilon}\kappa r_0 \sigma_0^{\tau+1} \exp\left(-B_0\, \sigma_j{}^{-\frac{1}{\rho-1}}\right)\, .$$

We will choose later $N = N(n,\tau,\rho)$ and $a > 0$ independent of κ, L_1, L_2, and r, so that $|P_0|_{\mathcal{U}_0} \leq \widetilde{\varepsilon}_0$ and $|P_j - P_{j-1}|_{\mathcal{U}_j} \leq \widetilde{\varepsilon}_j$ for $j \geq 1$. Now we put

$$E_j := c_1^{-1} \exp\left(-B\sigma_j{}^{-\frac{1}{\rho-1}}\right) \quad \text{with} \quad B := \frac{B_0}{2}\left(\delta^{-\frac{1}{\rho-1}} - 1\right),$$

where $c_1 > 1$ is the constant in the KAM step. We find $\delta \in (0,1)$ from the equalities

$$\forall j \in \mathbf{N}\ ,\quad E_{j+1} = c_1^{1/2} E_j^{3/2}\, .$$

This is equivalent to $\sigma_{j+1} = (2/3)^{\rho-1}\sigma_j$, and we get $\delta = \left(\dfrac{2}{3}\right)^{\rho-1}$, which implies $B = B_0/4 = A_0 L_1{}^{-\frac{1}{\rho-1}}$, where $A_0 = A_0(\rho) > 0$ depends only on ρ. Now we set $\eta_j = E_j^{1/2}$, $r_{j+1} = \eta_j r_j$, and put

$$\varepsilon_j = \widehat{\varepsilon}\kappa r_j \sigma_j^{\tau+1} E_j .$$

The choice of the 'weighted error' E_j above is motivated by the inequality $\widetilde{\varepsilon}_j \leq \varepsilon_{j+1}/2$, $j \geq 0$, which will be proved in (3.20). This inequality will allow us to put $P_j - P_{j-1}$ in the error term of the iteration of order j. Next we determine K_j from the equation $K_j^n e^{-K_j\sigma_j} = E_j$. Setting $x_j = K_j\sigma_j$ we obtain

$$x_j^n e^{-x_j} = E_j\sigma_j^n = c_1^{-1}\,\sigma_j^n\,\exp\left(-B{\sigma_j}^{-\frac{1}{\rho-1}}\right).$$

Consider the equation

$$x_j - n\ln x_j = B\sigma_j^{-\frac{1}{\rho-1}} - n\ln(\sigma_j) + \ln c_1. \tag{3.16}$$

We set

$$\sigma_0 \;=\; \sigma\, L_1^{-1}\,(\ln(L_1+e))^{-(\rho-1)}\;,\quad 0<\sigma\le\widetilde{\sigma}(n,\rho)\ll 1\,. \tag{3.17}$$

Obviously, $\sigma_0 L_1 < \sigma \le \widetilde{\sigma}(n,\rho) \ll 1$, and for any $j\in\mathbf{N}$ we obtain

$$B\sigma_j^{-\frac{1}{\rho-1}} - n\ln(\sigma_j) + \ln c_1 \ge B\sigma_0^{-\frac{1}{\rho-1}} = A_0(L_1\sigma_0)^{-\frac{1}{\rho-1}} > A_0\sigma^{-\frac{1}{\rho-1}} \gg 1\,.$$

Hence, choosing $0<\sigma\le\widetilde{\sigma}(n,\rho)\ll 1$, we obtain for each $j\in\mathbf{N}$ an unique solution $x_j = x_j(\sigma)$ of (3.16) such that

$$x_j \ge x_j - n\ln x_j \ge B\sigma_j^{-\frac{1}{\rho-1}} \ge A_0\sigma^{-\frac{1}{\rho-1}} \gg 1\,.$$

Then $x_j - n\ln x_j = x_j(1+o(1))$ as $\sigma\searrow 0$. On the other hand, using again (3.17) we get

$$\begin{aligned} x_j - n\ln x_j \;&\le\; B\sigma_j^{-\frac{1}{\rho-1}}\left[1 - nA_0^{-1}(L_1\sigma_j)^{\frac{1}{\rho-1}}\ln(L_1\sigma_j) + nA_0^{-1}(L_1\sigma_0)^{\frac{1}{\rho-1}}(\ln L_1 + \ln(c_1))\right] \\ &=\; B\sigma_j^{-\frac{1}{\rho-1}}(1+o(1))\,, \end{aligned}$$

uniformly with respect to $j\in\mathbf{N}$. Hence,

$$B\sigma_j^{-\frac{1}{\rho-1}} \le x_j \le B\sigma_j^{-\frac{1}{\rho-1}}(1+o(1))\,,\ \sigma\searrow 0, \tag{3.18}$$

uniformly with respect to $j\in\mathbf{N}$. We set $h_j = \kappa\,2^{-1}K_j^{-\tau-1}$ and fix $\upsilon = 1/54$.

We are going to check the hypothesis (a) and (b) in Proposition 3.2 for any $j\ge 0$ ((c) is fulfilled by definition). To prove (a) we use (3.15) and that $\eta_j^2 = E_j = o(1)$ as $\sigma\searrow 0$. Using (3.17) and (3.18) we obtain

$$\begin{aligned} \tfrac{\varepsilon_j}{r_j h_j} \;&=\; 2\widehat{\varepsilon}E_j {x_j}^{\tau+1} \;\le\; 2c_1^{-1}\exp\left(-B{\sigma_j}^{-\frac{1}{\rho-1}}\right)\left(B{\sigma_j}^{-\frac{1}{\rho-1}}\right)^{\tau+1}(1+o(1)) \\ &\le c(\rho,\tau)\left(-\tfrac{A_0}{2}(L_1\sigma_j)^{-\frac{1}{\rho-1}}\right) \le c(\rho,\tau)\left(-\tfrac{A_0}{2}(\sigma\delta^j)^{-\frac{1}{\rho-1}}\right). \end{aligned}$$

This implies $\varepsilon_j(r_jh_j)^{-1} \ll \mathrm{const.}\,\upsilon$ for $0<\sigma\le\widetilde{\sigma}(n,\rho,\tau)\ll 1$ which proves (b). In the same way we obtain

$$\prod_{j=0}^{\infty}\left(1+\frac{C\varepsilon_j}{r_jh_j}\right) \;\le\; \exp\left(\sum_{j=0}^{\infty}\frac{C\varepsilon_j}{r_jh_j}\right) \;\le\; 2 \tag{3.19}$$

for $0<\sigma\le\widetilde{\sigma}(n,\rho,\tau)\ll 1$, where $C = C(n,\tau)>0$ is the constant in Remark 3.3. We are going to check (3.12) with $\upsilon = 1/54$. Using (3.18) we obtain $x_j/x_{j+1} = (\sigma_{j+1}/\sigma_j)^{\frac{1}{\rho-1}}\,(1+o(1))$. This implies

$$\frac{h_{j+1}}{h_j} = \left(\frac{x_j}{x_{j+1}}\right)^{\tau+1}\left(\frac{\sigma_{j+1}}{\sigma_j}\right)^{\tau+1} = \delta^{(\tau+1)\frac{\rho}{\rho-1}}\,(1+o(1))) = \left(\frac{2}{3}\right)^{\rho(\tau+1)}(1+o(1))$$

for $\sigma \searrow 0$, uniformly with respect to $j \in \mathbf{N}$. Since $\rho > 1$ and $\tau + 1 > n \geq 2$, we obtain

$$\frac{h_{j+1}}{h_j} < \left(\frac{4}{9}\right)^{\rho} < \frac{4}{9} = \widetilde{\upsilon},$$

for any $0 < \sigma \leq \widetilde{\sigma}(n, \rho, \tau) \ll 1$, which proves (3.12).

Using the special choice of E_j, we are going to prove by induction that

$$\forall\, j \in \mathbf{N}\ , \quad \widetilde{\varepsilon}_j \ \leq \ \frac{1}{2}\varepsilon_{j+1}. \tag{3.20}$$

To obtain the estimate for $j = 0$ we write $\varepsilon_1 = \widehat{\varepsilon}\kappa r_1 \sigma_1^{\tau+1} E_1 = \widehat{\varepsilon}\kappa r_0 \sigma_0^{\tau+1}\delta^{\tau+1} c_1^{1/2} E_0^2$, and we obtain

$$\widetilde{\varepsilon}_0/\varepsilon_1 = c_1^{-1/2}\left(\frac{3}{2}\right)^{(\tau+1)(\rho-1)} \exp\left(-2B\sigma_0{}^{-\frac{1}{\rho-1}}\right) \leq 1/2$$

for $0 < \sigma \leq \widetilde{\sigma}(n, \rho, \tau) \ll 1$, since $B_0 = 4B$. To prove it for $j + 1 \geq 1$ we write

$$\varepsilon_{j+2} = \widehat{\varepsilon}\kappa r_{j+2}\sigma_{j+2}^{\tau+1} E_{j+2} = \widehat{\varepsilon}\kappa r_{j+1}\sigma_{j+1}^{\tau+1}\delta^{\tau+1} c_1^{1/2} E_{j+1}^2 = \varepsilon_{j+1} c_1^{1/2} E_{j+1}\delta^{\tau+1}.$$

Then for $j \geq 0$ we obtain

$$\frac{\widetilde{\varepsilon}_{j+1}}{\varepsilon_{j+2}}\left(\frac{\widetilde{\varepsilon}_j}{\varepsilon_{j+1}}\right)^{-1} = c_1^{-1/2}\left(\frac{3}{2}\right)^{(\tau+1)(\rho-1)} \exp\left(-\frac{1}{2}B\sigma_j{}^{-\frac{1}{\rho-1}}\right) \leq 1$$

for $0 < \sigma \leq \widetilde{\sigma}(n, \rho, \tau) \ll 1$ which implies by recurrence (3.20). From now on we fix $\sigma = \sigma(n, \rho, \tau) \ll 1$ so that all the estimates above hold and define σ_0 by (3.17). Then we set $s_0 = 5\sigma_0(1-\delta)^{-1}$. We are going to prove that the sequences $u_j = 4s_0\delta^j$, $v_j = 4r_0\delta^j$, and $w_j = 4h_0\delta^j$, verify (3.1) choosing $r_0 = cr$ and $c = c(n, \rho, \tau, \varsigma) \ll 1$. We have $4s_0 L_1 \leq 1$ in view of (3.17). Moreover, $h_0 < \kappa\sigma_0^{\tau+1} < \kappa s_0 \leq L_2^{-1-\varsigma} s_0$, and we obtain $w_j L_2 \leq u_j L_1$, and $w_j < L_2^{-1-\varsigma}$. Finally, $r_0 = cr \leq cL_2^{-1-\varsigma} < L_2^{-1-\varsigma}$, and $r_0 L_2 < cL_1^{-\varsigma} \leq s_0 L_1$, choosing appropriately $c = c(n, \rho, \tau, \varsigma) \ll 1$.

It remains to show that

$$|P_0|_{\mathcal{U}_0} \ \leq \ \widetilde{\varepsilon}_0\ , \quad |P_j - P_{j-1}|_{\mathcal{U}_j} \ \leq \ \widetilde{\varepsilon}_j,\ j \geq 1. \tag{3.21}$$

for $a \ll 1$. In view of (3.14) we have

$$|P_0|_{\mathcal{U}_0} \leq C_0\|P\|L_1^n = \widehat{\varepsilon}\kappa r_0 C_0 \frac{r}{r_0} L_1^{-N+n+2} a.$$

On the other hand, using (3.17) we get

$$\begin{aligned} \sigma_0^{\tau+1} E_0 \quad &= \quad c_1^{-1}\sigma_0^{\tau+1} \exp\left(-A_0\left(L_1\sigma_0\right)^{-\frac{1}{\rho-1}}\right) \\ &= \quad C'(n, \rho, \tau) L_1^{-\tau-1}(\ln(L_1 + e))^{-(\rho-1)(\tau+1)}(L_1 + e)^{-M} \geq C(n, \rho, \tau) L_1^{M-\tau-2}\,, \end{aligned}$$

where $M = A_0(\rho)\sigma(n, \rho, \tau)^{-1/(\rho-1)}$. Now we fix $N = M + \tau + n + 4$ and chose

$$a = C(n, \rho, \tau) C_0^{-1} \frac{r_0}{r} = C(n, \rho, \tau) c(n, \rho, \tau, \varsigma) C_0^{-1}.$$

Then $|P_0|_{\mathcal{U}_0} \leq \widetilde{\varepsilon}_0$. Recall that C_0 comes from the Approximation lemma and the extension in the beginning of Sect. 3, hence, a is independent of κ, L_1, L_2, and r. Using (3.14), we obtain for each $j \geq 1$

$$|P_j - P_{j-1}|_{\mathcal{U}_j} \leq \widetilde{\varepsilon}_0 \exp\left(-B_0\, {\sigma_j}^{-\frac{1}{\rho-1}}\right) \leq \widetilde{\varepsilon}_j.$$

We are ready to make the iterations. We consider the real-analytic in $\mathcal{U}_j$ Hamiltonian $H_j(\varphi, I;\omega) = N_0(I;\omega) + P_j(\varphi, I;\omega)$, where $N_0(I;\omega) := \langle\omega, I\rangle$. For any $j \in \mathbf{N}$, we denote by $\mathcal{D}_j$ the class of real-analytic diffeomorphisms $\mathcal{F}_j : D_{j+1} \times O_{j+1} \to D_j \times O_j$ of the form

$$\mathcal{F}_j(\theta, I;\omega) \;=\; (\Phi_j(\theta, I;\omega), \phi_j(\omega))\,,\;\; \Phi_j(\theta, I;\omega) \;=\; (U_j(\theta;\omega), V_j(\theta, I;\omega))\,, \tag{3.22}$$

where $\Phi_j(\theta, I;\omega)$ is affine linear with respect to I, and the transformation $\Phi_j(\,.\,,\,.\,;\omega)$ is canonical for any fixed ω. To simplify the notations we denote the sup-norm in $D_j \times O_j$ by $|\cdot|_j$ instead of $|\cdot|_{s_j,r_j,h_j}$. Obviously, $D_j \times O_j \subset \mathcal{U}_j$.

Proposition 3.5 *Suppose P_j, $j \geq 0$, is real-analytic on $\mathcal{U}_j$ with*

$$|P_0|_{\mathcal{U}_0} \;\leq\; \widetilde{\varepsilon}_0\;, \quad |P_j - P_{j-1}|_{\mathcal{U}_j} \;\leq\; \widetilde{\varepsilon}_j,\;\; j \geq 1.$$

Then for each $j \geq 0$ there exists a real analytic normal form $N_j(I;\omega) = e_j(\omega) + \langle\omega, I\rangle$ and a real analytic transformation $\mathcal{F}^j$, where $\mathcal{F}^0 = \mathrm{Id}$ and

$$\mathcal{F}^{j+1} = \mathcal{F}_0 \circ \cdots \circ \mathcal{F}_j :\; D_{j+1} \times O_{j+1} \longrightarrow (D_0 \times O_0) \cap \mathcal{U}_j,\;\; j \geq 0,$$

with $\mathcal{F}_j \in \mathcal{D}_j$ such that $H_j \circ \mathcal{F}^{j+1} = N_{j+1} + R_{j+1}$ and $|R_{j+1}|_{j+1} \leq \varepsilon_{j+1}$. Moreover,

$$|\overline{W}_j(\mathcal{F}_j - \mathrm{id})|_{j+1}\;,\;\; |\overline{W}_j(D\mathcal{F}_j - \mathrm{Id})\overline{W}_j^{-1}|_{j+1} \;<\; \frac{C\varepsilon_j}{r_j h_j}, \tag{3.23}$$

$$|\overline{W}_0(\mathcal{F}^{j+1} - \mathcal{F}^j)|_{j+1} \;<\; \frac{c\varepsilon_j}{r_j h_j}, \tag{3.24}$$

where $C = C(n,\rho) > 0$ is the constant in Remark 3.3, $c = c(n,\rho) > 0$, $D\mathcal{F}^j$ stands for the Jacobian of $\mathcal{F}^j$ with respect to (θ, I, ω), and $\overline{W}_j = \mathrm{diag}\left({\sigma_j}^{-1}\mathrm{Id}, {r_j}^{-1}\mathrm{Id}, {h_j}^{-1}\mathrm{Id}\right)$.

Proof. The proof is similar to that of the Iterative Lemma [9]. First, applying the KAM step we find $\mathcal{F}^1 = \mathcal{F}_0$ such that $H_0 \circ \mathcal{F}_0 = N_1 + R_1$, where R_1 is real analytic in $D_1 \times O_1$ and $|R_1|_1 \leq \varepsilon_1$. By recurrence we define for any $j \geq 1$ the transformation $\mathcal{F}^{j+1} = \mathcal{F}^j \circ \mathcal{F}_j$, where $\mathcal{F}_j$ belongs to $\mathcal{D}_j$. By the inductive assumption we have $H_{j-1} \circ \mathcal{F}^j = N_j + R_j$, where $N_j(I;\omega) = e_j(\omega) + \langle\omega, I\rangle$ is a real-analytic normal form, R_j is real analytic in $D_j \times O_j$, and $|R_j|_j \leq \varepsilon_j$. Then we write

$$\begin{aligned}
H_j \circ \mathcal{F}^{j+1} &= (N_0 + P_{j-1}) \circ \mathcal{F}^{j+1} + (P_j - P_{j-1}) \circ \mathcal{F}^{j+1} \\
&= \left[H_{j-1} \circ \mathcal{F}^j\right] \circ \mathcal{F}_j + (P_j - P_{j-1}) \circ \mathcal{F}^{j+1} \\
&= (N_j + R_j) \circ \mathcal{F}_j + (P_j - P_{j-1}) \circ \mathcal{F}^{j+1}.
\end{aligned}$$

We apply Proposition 3.2 to the Hamiltonian $N_j + R_j$ which is real-analytic in $D_j \times O_j$. In this way, using (3.13), we find a real-analytic map $\mathcal{F}_j : D_{j+1} \times O_{j+1} \to D_j \times O_j$ which belongs to the class $\mathcal{D}_j$ and such that $(N_j + R_j) \circ \mathcal{F}_j = N_{j+1} + R_{j+1,1}$, where

$$|R_{j+1,1}|_{j+1} \leq \frac{1}{2}\widehat{\varepsilon}\kappa r_{j+1}\sigma_{j+1}^{\tau+1} c_1^{1/2} E_j^{3/2} = \frac{\varepsilon_{j+1}}{2}.$$

Moreover, $\mathcal{F}_j$ satisfies (3.23) in view of Remark 3.3 and as in [9] we obtain (3.24). We are going to show that

$$\mathcal{F}^{j+1} : D_{j+1} \times O_{j+1} \ \longrightarrow \ \mathcal{U}_j \, . \tag{3.25}$$

This inequality combined with (3.20) and (3.21) implies

$$|(P_j - P_{j-1}) \circ \mathcal{F}^{j+1}|_{j+1} \leq |P_j - P_{j-1}|_{\mathcal{U}_j} \leq \widetilde{\varepsilon}_j \leq \frac{\varepsilon_{j+1}}{2},$$

and we obtain $H_j \circ \mathcal{F}^{j+1} = N_{j+1} + R_{j+1}$, where $|R_{j+1}|_{j+1} \leq \varepsilon_{j+1}$.

To prove (3.25) we note that

$$|\overline{W}_k \overline{W}_{k+1}^{-1}| = \sup\{s_{k+1}/s_k, r_{k+1}/r_k, h_{k+1}/h_k\} = s_{k+1}/s_k = \delta,$$

since $r_{k+1}/r_k = E_k^{1/2} \ll \delta$, and $h_{k+1}/h_k \sim \delta^{(\tau+1)\rho(\rho-1)^{-1}} < \delta$ for any $k \in \mathbf{N}$. Then using (3.19) and (3.23), we estimate the Jacobian of $\mathcal{F}^{j+1}$ in $D_{j+1} \times O_{j+1}$ as follows (see [9])

$$\begin{aligned}
\left|\overline{W}_0 D\mathcal{F}^{j+1}\overline{W}_j^{-1}\right|_{j+1} &= \left|\overline{W}_0 D(\mathcal{F}_0 \circ \cdots \circ \mathcal{F}_j)\overline{W}_j^{-1}\right|_{j+1} \\
&\leq \textstyle\prod_{k=0}^{j-1}\left(\left|\overline{W}_k D\mathcal{F}_k \overline{W}_k^{-1}\right|_{k+1} \left|\overline{W}_k \overline{W}_{k+1}^{-1}\right|\right)\left|\overline{W}_j D\mathcal{F}_j \overline{W}_j^{-1}\right|_{j+1} \\
&\leq \delta^j \textstyle\prod_{j=0}^{\infty}\left(1 + \frac{C\varepsilon_j}{r_j h_j}\right) \ \leq \ 2\,\delta^j \, .
\end{aligned}$$

Set $z = (\theta, I, \omega) = x + iy \in D_{j+1} \times O_{j+1}$, where x and y are the real and the imaginary part of z. Then

$$\mathcal{F}^{j+1}(x+iy) \ = \ \mathcal{F}^{j+1}(x) + \overline{W}_0^{-1} T_{j+1}(x,y)\overline{W}_j\, y\,,$$

$$T_{j+1}(x,y) = i\int_0^1 \overline{W}_0 D\mathcal{F}^{j+1}(x+ity)\overline{W}_j^{-1}\, dt\,.$$

Moreover, $|T_{j+1}(x,y)| \leq 2\,\delta^j$ and using that $|\overline{W}_j\, y| \leq |\overline{W}_{j+1}\, y| \leq \sqrt{3}$ we get

$$|T_{j+1}(x,y)\overline{W}_j\, y| \leq 4\,\delta^j\,, \quad x+iy \in D_{j+1} \times O_{j+1}\,.$$

This implies $\mathcal{F}^{j+1}(x+iy) \in \mathcal{U}_j$, since $\mathcal{F}^{j+1}(x)$ is real, and we complete the proof of Proposition 3.5. □

We are going to prove suitable Gevrey estimates for $\mathcal{F}^j$. We set

$$\widetilde{D}_j = \{(\theta, I) \in D_j : \ |\mathrm{Im}\,\theta| < s_j/2\}\ , \quad \widetilde{O}_j = \{\omega \in \mathbf{C}^n : \ |\omega - \Omega_\kappa| < h_j/2\}\,,$$

and we denote $\mathcal{S}^j = \mathcal{F}^{j+1} - \mathcal{F}^j$. For any multi-indices α and β and $m \in \mathbf{N}$ with $|\beta| \leq m$, we denote

$$R^m_{\omega'}\left(\partial_\theta^\alpha \partial_\omega^\beta \mathcal{S}^j\right)(\theta, I, \omega) := \partial_\theta^\alpha \partial_\omega^\beta \mathcal{S}^j(\theta, I, \omega) - \sum_{|\beta+\gamma| \leq m} (\omega - \omega')^\gamma \partial_\theta^\alpha \partial_\omega^{\beta+\gamma} \mathcal{S}^j(\theta, I, \omega')/\gamma!.$$

Recall that $\rho' = \rho(\tau+1) + 1$.

Lemma 3.6 *Under the assumptions of Proposition 3.5 we have*

$$
\begin{aligned}
|\overline{W}_0 \partial_\theta^\alpha \partial_\omega^\beta \mathcal{S}^j(\theta,0,\omega)| \quad &\leq \quad \widehat{\varepsilon}\, A\, C^{|\alpha+\beta|}\, L_1^{|\alpha|+|\beta|(\tau+1)+1} \kappa^{-|\beta|}\, \alpha!^{\,\rho} \beta!^{\,\rho'} E_j^{1/2}, \\
& \qquad (\theta,0,\omega) \in \widetilde{D}_{j+1} \times \widetilde{O}_{j+1}\,, \\
|\overline{W}_0 R^m_{\omega'}(\partial_\theta^\alpha \partial_\omega^\beta \mathcal{S}^j(\theta,0,\omega))| \quad &\leq \quad \widehat{\varepsilon}\, A\, C^{m+|\alpha|+1} L_1^{|\alpha|+(m+1)(\tau+1)+1} \kappa^{-m-1} \\
& \times \quad \tfrac{|\omega-\omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!}\, \alpha!^{\,\rho}\, (m+1)!^{\,\rho'} E_j^{1/2}\,, \quad \theta \in \mathbf{T}^n,\ \omega,\omega' \in \Omega_\kappa\,,
\end{aligned}
$$

for any $m \in \mathbf{N}$, $\alpha,\beta \in \mathbf{N}^n$, $|\beta| \leq m$, where the constants A, C depend only on τ, ρ, n and ς.

Proof. Using (3.24) and the Cauchy estimate, we evaluate $\partial_\theta^\alpha \partial_\omega^\beta \mathcal{S}^j$ for any $j \geq 0$ and $|\alpha+\beta| \geq 1$ in $\widetilde{D}_{j+1} \times \widetilde{O}_{j+1}$. We have

$$
M_{j,\alpha,\beta} := \left| W_0 \partial_\theta^\alpha \partial_\omega^\beta \mathcal{S}^j \right| \leq c \frac{2^{|\alpha+\beta|}\, \alpha!\, \beta!\, \varepsilon_j}{r_j h_j s_{j+1}^{|\alpha|} h_{j+1}^{|\beta|}} = c\kappa\widehat{\varepsilon} \frac{2^{|\alpha+\beta|}\, \alpha!\, \beta!\, E_j \sigma_j^\tau}{h_j s_{j+1}^{|\alpha|} h_{j+1}^{|\beta|}}.
$$

Recall that $s_j = 5(1-\delta)^{-1}\sigma_j = 5(1-\delta)^{-1} A_0^{\rho-1} L_1^{-1} (B\sigma_j^{-\frac{1}{\rho-1}})^{-(\rho-1)}$, and

$$
h_j = \frac{\kappa}{2} K_j^{-\tau-1} = \frac{\kappa}{2} \sigma_j^{\tau+1} x_j^{-\tau-1}\,.
$$

Then by (3.18) we get

$$
h_{j+1}^{-1} \leq \kappa^{-1} \widetilde{C}_0 L_1^{\tau+1} \left(B\sigma_j^{-\frac{1}{\rho-1}} \right)^{\rho(\tau+1)}.
$$

where $\widetilde{C}_0$ depends only on τ and ρ. This implies

$$
\begin{aligned}
M_{j,\alpha,\beta} \quad &\leq \quad \widehat{\varepsilon}\, A_1 C_1^{|\alpha+\beta|} L_1^{|\alpha|+|\beta|(\tau+1)+1} \kappa^{-|\beta|} \alpha!\, \beta! \\
& \times \quad \left(B\sigma_j^{-\frac{1}{\rho-1}} \right)^{(\rho-1)(|\alpha|-\tau)+\rho(\tau+1)(|\beta|+1)} \exp\left(-B\sigma_j^{-\frac{1}{\rho-1}} \right),
\end{aligned}
$$

where A_1, C_1 depend only on τ and ρ. Then we obtain

$$
M_{j,\alpha,\beta} \leq \widehat{\varepsilon} A\, C^{|\alpha+\beta|} L_1^{|\alpha|+|\beta|(\tau+1)+1} \kappa^{-|\beta|} \alpha!^{\,\rho}\, \beta!^{\,\rho(\tau+1)+1}\, E_j^{1/2}\,. \tag{3.26}
$$

where A, C depend only on τ and ρ.

We are going to prove the second estimate for $\omega, \omega' \in \Omega_\kappa$. First we suppose that $|\omega' - \omega| \leq h_{j+1}/8$. Expanding the analytic in O_{j+1} function $\omega \to \partial_\omega^\alpha \mathcal{S}^j(\theta,0,\omega)$, $\theta \in \mathbf{T}^n$, in Taylor series with respect to ω at ω', and using as above the Cauchy estimate for $M_{j,\alpha,\gamma}$, we evaluate

$$
L^m_{j,\alpha,\beta} := |W_0 (R^m_{\omega'}\, \partial_\theta^\alpha \partial_\omega^\beta \mathcal{S}^j)(\theta,0;\omega))|\,, \quad \theta \in \mathbf{T}^n,\ \omega,\, \omega' \in \Omega_\kappa.
$$

For $|\beta| \leq m+1$ we have

$$
\frac{(\beta+\gamma)!}{\gamma!} \leq 2^{|\beta+\gamma|} \beta! \leq 2^{|\beta+\gamma|} \frac{(m+1)!}{(m-|\beta|+1)!}.
$$

Then we obtain as above

$$\begin{aligned} L^m_{j,\alpha,\beta} &\le \sum_{|\gamma|\ge m-|\beta|+1} |\omega'-\omega|^{|\gamma|}\, M_{j,\alpha,\beta+\gamma}(\theta,0,\omega')/\gamma! \\ &\le c\,\alpha!(m+1)!\,\frac{|\omega'-\omega|^{m-|\beta|+1}}{(m-|\beta|+1)!}\,\frac{4^{|\alpha|+m+1}\varepsilon_j}{r_j h_j s_{j+1}^{|\alpha|} h_{j+1}^{m+1}} \sum_{|\beta+\gamma|\ge m+1} \left(4|\omega'-\omega|h_{j+1}^{-1}\right)^{|\beta+\gamma|-m-1}, \end{aligned}$$

and we get

$$\begin{aligned} L^m_{j,\alpha,\beta} &\le 2c\,\alpha!(m+1)!\,\frac{|\omega'-\omega|^{m-|\beta|+1}}{(m-|\beta|+1)!}\,\frac{4^{|\alpha|+m+1}\varepsilon_j}{r_j h_j s_{j+1}^{|\alpha|} h_{j+1}^{m+1}} \\ &\le \widehat{\varepsilon}\,A\,C^{|\alpha|+m+1} L_1^{|\alpha|+(m+1)(\tau+1)+1}\kappa^{-m-1}\,\frac{|\omega'-\omega|^{m-|\beta|+1}}{(m-|\beta|+1)!}\,\alpha!^{\rho}(m+1)!^{\,\rho(\tau+1)+1} E_j^{1/2}. \end{aligned}$$

where A, C depend only on τ, ρ and n. For $|\omega'-\omega|\ge h_{j+1}/8$ we obtain the same inequality, estimating $L^m_{j,\alpha,\beta}$ term by term and using (3.26). This proves the lemma. □

According to Proposition 3.5 and Lemma 3.6, the limit

$$\partial_\theta^\alpha \mathcal{H}^\beta(\theta,\omega) := \lim_{j\to\infty} \partial_\theta^\alpha \partial_\omega^\beta \left[\mathcal{F}^j(\theta,0,\omega)-(\theta,0,\omega)\right], \quad (\theta,\omega)\in \mathbf{T}^n\times\Omega_\kappa,$$

exists for each $\alpha,\beta\in\mathbf{N}$ and it is uniform since $\sum E_j^{1/2}<\infty$. Moreover, the partial derivatives of $\partial_\theta^\alpha(\mathcal{H}^\beta)=\partial_\theta^\alpha\mathcal{H}^\beta$ exist and they are continuous on $\mathbf{T}^n\times\Omega_\kappa$. Consider the jet $\mathcal{H}=\left(\partial_\theta^\alpha\mathcal{H}^\beta\right)$, $\alpha,\beta\in\mathbf{N}^n$, of continuous functions $\partial_\theta^\alpha\mathcal{H}^\beta:\mathbf{T}^n\times\Omega_\kappa\to\mathbf{T}^n\times D\times\Omega$, and set

$$\left(R^m_{\omega'}\partial_\theta^\alpha\mathcal{H}\right)_\beta(\theta,\omega) := \partial_\theta^\alpha\mathcal{H}^\beta(\theta,\omega) - \sum_{|\beta+\gamma|\le m} (\omega-\omega')^\gamma \partial_\theta^\alpha\mathcal{H}^{\beta+\gamma}(\theta,\omega')/\gamma!.$$

In view of Lemma 3.6, we have

$$\begin{aligned} |\overline{W}_0\partial_\theta^\alpha\mathcal{H}^\beta(\theta,\omega)| &\le \widehat{\varepsilon}\,A\,L_1\,(CL_1)^{|\alpha|}\,(CL_1^{\tau+1}\kappa^{-1})^{|\beta|}\alpha!^{\,\rho}\beta!^{\,\rho'} \\ |\overline{W}_0\left(R^m_{\omega'}\partial_\theta^\alpha\mathcal{H}\right)_\beta(\theta,\omega)| &\le \widehat{\varepsilon}\,A\,L_1\,(CL_1)^{|\alpha|}\,(CL_1^{\tau+1}\kappa^{-1})^{m+1}\frac{|\omega-\omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!}\,\alpha!^{\,\rho}\,(m+1)!^{\,\rho'} \end{aligned} \tag{3.27}$$

for each α, and β satisfying $0\le|\beta|\le m$, and $\theta\in\mathbf{T}^n$, $\omega,\omega'\in\Omega_\kappa$, where A and C depend only on τ, ρ, ρ', and n. We are going to extend $\mathcal{H}$ to a Gevrey function on $\mathbf{T}^n\times\Omega$.

3.4 *Whitney extension in Gevrey classes*

Let K be a compact in $\mathbf{R}^n$ and $\rho\ge 1$, $\rho'>1$. We consider a jet (f^β), $\beta\in\mathbf{N}^n$, of functions $f^\beta:\mathbf{T}^n\times K\to\mathbf{R}$, such that for each $\alpha\in\mathbf{N}^n$ the partial derivative $\partial_\theta^\alpha f^\beta$ exists, it is continuous on $\mathbf{T}^n\times K$, and there are positive constants A, C_1 and C_2 such that

$$\begin{aligned} |\partial_\theta^\alpha f^\beta(\theta,\omega)| &\le A\,C_1^{|\alpha|}C_2^{|\beta|}\,\alpha!^{\,\rho}\beta!^{\,\rho'}, \\ |\left(R^m_{\omega'}\partial_\theta^\alpha f\right)_\beta(\theta,\omega)| &\le A\,C_1^{|\alpha|}C_2^{m+1}\,\frac{|\omega-\omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!}\,\alpha!^{\,\rho}\,(m+1)!^{\,\rho'}. \end{aligned} \tag{3.28}$$

Theorem 3.7 *There exist positive constants A_0 and C_0 and for any compact set K and any jet $f = (f^\beta)$, $\beta \in \mathbf{N}^n$, satisfying (3.28) there exists $\widetilde{f} \in \mathcal{G}^{\rho,\rho'}(\mathbf{T}^n \times \mathbf{R}^n)$ such that $\partial_\theta^\alpha \partial_\omega^\beta \widetilde{f} = \partial_\theta^\alpha f^\beta$ on K for each α, β, and*

$$|\partial_\theta^\alpha \partial_\omega^\beta \widetilde{f}(\theta,\omega)| \;\le\; A\, A_0 \,\max(C_1,1)\,(C_0C_1)^{|\alpha|+1}\,(C_0C_2)^{|\beta|}\,\alpha!^{\,\rho}\beta!^{\,\rho'}\,.$$

Remark. We point out that the positive constants A_0 and C_0 do not depend on the the jet f, on compact set K nor on the constants A, C_1 and C_2.

Proof. Consider the Fourier coefficients

$$f_k^\beta(\omega) \;=\; (2\pi)^{-n}\int_{\mathbf{T}^n} e^{-i\langle k,\varphi\rangle}\, f^\beta(\varphi,\omega)d\varphi\ ,\ k\in\mathbf{Z}^n\ ,$$

and denote by $\mathcal{A}$ the set of all Whitney jets $g_k = \left(g_k^\beta\right)$, $\beta\in\mathbf{N}^n$, where $g_k^\beta = e^{r|k|^{1/\rho}}\, f_k^\beta(\omega)$ and $k\in\mathbf{Z}^n$, $|k| = |k_1|+\cdots+|k_n|$. Choosing $r = c_0C_1^{-1/\rho}$ with $0 < c_0 = c_0(n,\rho) \ll 1$, we are going to show that $(g_k^\beta)\in\mathcal{A}$ satisfy

$$\begin{aligned}
\left|g_k^\beta(\omega)\right| \;&\le\; A_2C_2^{|\beta|}\,\beta!^{\,\rho}\ ,\ \omega\in K\\
|\left(R_{\omega'}^m g_k\right)_\beta(\omega)| \;&\le\; A_2C_2^{m+1}\,\frac{|\omega-\omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!}\,(m+1)!^{\,\rho'}\,,
\end{aligned}\tag{3.29}$$

where $A_2 = 2A\max(C_1,1)$. We decompose $j = \rho\ell_j + q_j$, $j,\ell\in\mathbf{N}$, $0\le q_j<\rho$. For any $k = (k_1,\ldots,k_n)\in\mathbf{Z}^n$ we have $|k|\le n\max_{1\le i\le n}|k_i| = n|k_p|$ for some p. Then integrating by parts and using (3.28), we estimate

$$\frac{r^j|k|^{j/\rho}}{j!}|f_k^\beta(\omega)| \le A\frac{r^j}{j!}n^{\ell_j+1}C_1^{\ell_j+1}C_2^{|\beta|}(\ell_j+1)!^\rho\beta!^{\,\rho'} \le 2^{-j}A\max(C_1,1)\,C_2^{|\beta|}\beta!^{\,\rho'},$$

choosing $r = c_0C_1^{-1/\rho}$ with $0<c_0 = c_0(n,\rho)\ll 1$. This proves the first part of (3.29). To prove the second part, we notice that $(R_{\omega'}^m f_k)_\beta(\omega)$ is just the Fourier coefficient of $(R_{\omega'}^m\partial_\theta^\alpha f)_\beta(\theta,\omega)$ corresponding to k. Now we use a variant of the Whitney extension theorem due to Bruna (see Theorem 3.1, [1]).

Theorem 3.8 *For any compact set K and a jet $g = (g^\beta)$, $\beta\in\mathbf{N}^n$, satisfying (3.29) on K, there is $\widetilde{g}\in\mathcal{G}^{\rho'}(\mathbf{R}^n)$ such that $\partial_\omega^\beta\widetilde{g} = g^\beta$ on K for any β, and*

$$|\partial_\omega^\beta\widetilde{g}(\omega)| \;\le\; A_0A_2\,(C_0C_2)^{|\beta|}\,\beta!^{\,\rho'}\,.$$

Moreover, the positive constants A_0 and C_0 do not depend on the jet g, on the compact set K nor on the constants A_2, C_2.

The proof of Theorem 3.8 is given in [1]. Here we only indicate that the constants A_0, $C_0>0$ do not depend on the compact set K nor on A_2 and C_2. This follows from the proof of Theorem 3.1, [1]. More precisely, setting $f^\beta(\omega) = A_2^{-1}C_2^{-|\beta|}g^\beta(C_2^{-1}\omega)$, $C_2^{-1}\omega\in K$, we obtain $|f^\beta(\omega)|\le\beta!^{\,\rho'}$ and

$$|\left(R_{\omega'}^m f\right)_\beta(\omega)| \;=\; A_2^{-1}C_2^{-|\beta|}\left|\left(R_{C_2^{-1}\omega'}^m g\right)_\beta(C_2^{-1}\omega)\right| \;\le\; \frac{|\omega-\omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!}\,(m+1)!^{\,\rho'}$$

for $C_2^{-1}\omega, C_2^{-1}\omega' \in K$. Hence, we can suppose that (12) and (13), [1], hold with $\varepsilon = A = 1$ (K is scaled to another compact still denoted by K). Then it is easy to see that the constants A and ε in (19), [1], are independent of g and K. Moreover, the different constants in Lemma 3.2 and 3.3, [1], do not depend on g and K, and we obtain that the constants in (26), [1], are independent of g and K. Scaling back by C_2 we obtain the desired estimates with constants A_0 and C_0 independent of g, K, A_2 and C_2. $\square$

Applying Theorem 3.8 to the family of jets $\mathcal{A}$, we obtain a family of functions $\widetilde{g}_k \in \mathcal{G}^{\rho'}$ such that $\partial_\omega^\beta \widetilde{g}_k = g_k^\beta$ on K for each β, and

$$\left|\partial_\omega^\beta \, \widetilde{g}_k(\omega)\right| \ \le A_0 A \max(C_1, 1)(C_0C_2)^{|\beta|}\, \beta!^{\,\rho'} \ , \ \forall\, \omega \in \mathbf{R}^n \, , \ k \in \mathbf{Z}^n \, , \ \beta \in \mathbf{N}^n \, .$$

Now it is easy to see that the function

$$\widetilde{f}(\theta,\omega) \ = \ \sum_{k\in\mathbf{Z}^n} e^{i\langle k,\theta\rangle - r|k|^{1/\rho}}\, \widetilde{g}_k(\omega)$$

satisfies the requirements of Theorem 3.8. $\square$

3.5 *Proof of Theorem 2.1.*

Using (3.27), we extend the jet $\mathcal{H}$ to a Gevrey function $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) : \mathbf{T}^n \times \Omega \to \mathbf{T}^n \times D \times \Omega$. We have

$$|\overline{W}_0 \partial_\theta^\alpha \partial_\omega^\beta \mathcal{H}(\theta,\omega)| \ \le \ \widehat{\varepsilon}\, A\, L_1^2\, (CL_1)^{|\alpha|}\, (CL_1^{\tau+1}\kappa^{-1})^{|\beta|} \alpha!^{\,\rho} \beta!^{\,\rho(\tau'+1)+1} \, ,$$

where $\widehat{\varepsilon} = \|P\| L_1^{N-2}(a\kappa r)^{-1}$ and the positive constants A and C are independent of L_1, L_2, κ, r, and $\Omega \subset B_{\bar{R}}(0)$. We set $\mathcal{F} = (\Phi, \phi)$, $\Phi = (U, V)$, where $U(\theta,\omega) = \mathcal{H}_1(\theta,\omega) + \theta$, $V(\theta,\omega) = \mathcal{H}_2(\theta,\omega)$, and $\phi(\omega) = \mathcal{H}_3(\omega) + \omega$. Recall that $r_0 = cr$, where $c = c(n,\tau,\rho,\varsigma) > 0$ is fixed in Sect. 3.3. On the other hand, $h_0 \le \kappa\sigma_0^{\tau+1} < \kappa$, and we obtain

$$\begin{aligned} &\left|\partial_\theta^\alpha \partial_\omega^\beta (U(\theta;\omega) - \theta)\right| \ + \ r^{-1} \left|\partial_\theta^\alpha \partial_\omega^\beta V(\theta;\omega)\right| + \kappa^{-1} \left|\partial_\omega^\beta (\phi(\omega) - \omega)\right| \\ &\le \ A \frac{\|P\| L_1^N}{\kappa r} \, (CL_1)^{|\alpha|}\, (CL_1^{\tau+1}\kappa^{-1})^{|\beta|} \alpha!^{\,\rho} \beta!^{\,\rho'} \end{aligned}$$

for some positive constants A and C as above. Choosing $\epsilon < 1/A$ in (2.3) we obtain $|V(\theta,\omega)| \le A\|P\| L_1^N \kappa^{-1} \le A\epsilon r < r \le R$. In the same way we get $\phi(\omega) \in \Omega$ for $\omega \in \Omega_\kappa$. This proves the estimates in Theorem 2.1. As in Sect. 5.d, [9], we obtain that $\left|X_{H_j} \circ \mathcal{F}^j - D\Phi^j \cdot X_N\right| \le \frac{c\varepsilon_j}{r_j h_j}$ on $\mathbf{T}^n \times \{0\} \times \Omega_\kappa$ for all $j \ge 0$, where X_{H_j} and X_N stand for the Hamiltonian vector fields of $H_j(\theta, I; \omega)$ and $N = \langle \omega, I\rangle$, respectively. On the other hand, ∇H_j converges uniformly to ∇H as $j \to \infty$ in view of Proposition 3.1, hence,

$$X_{H(\cdot;\phi(\omega))} \circ \Phi = D\Phi \cdot X_N$$

on $\mathbf{T}^n \times \{0\} \times \Omega_\kappa$. Then $\{\Phi(\theta;\omega) : \theta \in \mathbf{T}^n\}$ is an embedded invariant torus of the Hamiltonian $H(\theta, I; \phi(\omega))$ with frequency $\omega \in \Omega_\kappa$. It is Lagrangian by construction (see also [2], Sect. I.3.2). This completes the proof of Theorem 1.1. $\square$

3.6 *Real analytic hamiltonians.*

Consider a real analytic Hamiltonian $P \in \mathcal{G}^1_{L_1,L_2,L_2}(\mathbf{T}^n \times B \times \Omega)$, $B = B_R(0)$, with norm

$$\|P\|_{L_1,L_2} = \sup \left(|\partial_\theta^\alpha \partial_I^\beta \partial_\omega^\gamma P(\theta, I; \omega)| L_1^{-|\alpha|} L_2^{-|\beta|-|\gamma|} (\alpha!\beta!\gamma!)^{-1} \right) < +\infty.$$

Then P can be extended as an analytic Hamiltonian in $D_{s,r} \times O_h$, where $s = (2L_1)^{-1}$ and $r = h = (2L_2)^{-1}$, and with sup-norm satisfying $\|P\|_{2L_1,2L_2} \le \|P\|_{s,r,h} \le C\|P\|_{L_1,L_2}$, where $C = C(n) > 0$. Fix $\tau' > \tau > n-1$. We can slightly improve Theorem 3.1 [10]. Given $s > 0$ we denote by $\mathcal{U}_s$ the set of all $\theta \in \mathbf{C}^n/2\pi\mathbf{Z}^n$ such that $|\mathrm{Im}\,\theta| < s$.

Theorem 3.9 *Suppose that H is given by (2.1) where $P \in \mathcal{G}^1_{L_1,L_2,L_2}(\mathbf{T}^n \times B_R(0) \times \Omega)$. Fix $\kappa > 0$ and $r > 0$ such that $\kappa, r < L_2^{-1}$ and $r \le R$. Then there is $0 < s_0 \le s$ and $\epsilon > 0$ both independent of κ, L_2, r, R, and of $\Omega \subset \Omega_0$, such that if $\|P\| \le \epsilon\kappa r$ then there exist maps $\phi \in \mathcal{G}^{\tau'+2}(\Omega,\Omega)$ and $\Phi = (U,V) \in \mathcal{G}^{1,\tau'+2}(\mathcal{U}_{s_0/2} \times \Omega, \mathcal{U}_{s_0} \times B_R(0))$, satisfying (i), Theorem 2.1, with $\rho = 1$. Moreover, there exist $A, C > 0$, independent of κ, L_2, r, and Ω, such that*

$$\left|\partial_\omega^\beta (U(\theta;\omega) - \theta)\right| + r^{-1}\left|\partial_\omega^\beta V(\theta;\omega)\right| + \kappa^{-1}\left|\partial_\omega^\beta(\phi(\omega) - \omega)\right| \le A\, C^{|\beta|}\, \kappa^{-|\beta|}\, \beta!^{\,\tau'+2}\, \frac{\|P\|}{\kappa r}$$

uniformly in $(\theta,\omega) \in \mathcal{U}_{s_0/2} \times \Omega$ and for any $\beta \in \mathbf{N}^n$.

Proof. Fix $\rho = (\tau' - \tau)(\tau+1)^{-1} + 1$ and set as above $\sigma_j = \sigma_0\delta^j$, and $s_{j+1} = s_j - 5\sigma_j$, $j \ge 0$, where $s_0(1-\delta) = 20\sigma_0$ (then $s_j \to 3s_0/4$) and $s_0 \le s = (2L_1)^{-1}$. As above we define $E_j := c_1^{-1}\exp(-\sigma_j^{-\frac{1}{\rho-1}})$ and set $\varepsilon_j = \widehat{\varepsilon}\kappa r_j \sigma_j^{\tau+1} E_j$, where $\widehat{\varepsilon} = \|P\|_{s,r,h}(a\kappa r)^{-1}$. Choose $r_0 = r$ and $h_0 = h$ and define r_j, x_j, K_j and h_j as above. We consider $H = \langle \omega, I\rangle + P$ in $D_j \times O_j$. We fix $\sigma_0 = \sigma_0(n,\rho,\tau,s) \le s(1-\delta)/20$, so that a), b), c) in Proposition 3.1 and (3.12) hold for any $j \ge 0$. Next we choose $a = a(n,\rho,\tau,L_1) \le c_1^{-1}\sigma_0^{-\tau-1}\exp(-\sigma_0^{-\frac{1}{\rho-1}})$. Then $|P|_0 \le \varepsilon_0$, and we can apply Proposition 3.5. Moreover, as in Lemma 3.6 we obtain with $\rho' = \rho(\tau+1)+1 = \tau'+2$

$$\begin{aligned}
|\overline{W}_0 \partial_\omega^\beta \mathcal{S}^j(\theta,0,\omega)| &\le \widehat{\varepsilon}\, A\, C^{|\beta|}\, \kappa^{-|\beta|}\, \beta!^{\,\rho'} E_j^{1/2}, \ (\theta,0,\omega) \in \mathcal{U}_{s_0/2} \times \widetilde{O}_{j+1}\,, \\
|\overline{W}_0 R^m_{\omega'}(\partial_\omega^\beta \mathcal{S}^j(\theta,0,\omega))| &\le \widehat{\varepsilon}\, A\, C^{m+1}\kappa^{-m-1} \\
&\times \frac{|\omega-\omega'|^{m-|\beta|+1}}{(m-|\beta|+1)!}\,(m+1)!^{\,\rho'} E_j^{1/2}\,, \quad \theta \in \mathcal{U}_{s_0/2},\ \omega,\omega' \in \Omega_\kappa\,,
\end{aligned}$$

for any $m \in \mathbf{N}$, $\beta \in \mathbf{N}^n$, $|\beta| \le m$, where the constants A, C depend only on τ, ρ, n and L_1. We complete the proof of the theorem as above. □

4 *Proof of Theorem 1.1 and Corollary 1.2.*

Proof of Theorem 1.1. Set $r = R = \kappa\sqrt{\epsilon_H}$. Then (2.2) implies $\|P\| \le (A+1)\kappa r\sqrt{\epsilon_H}$ and we can apply Theorem 2.1. Consider the map $\overline{\Phi} : \mathbf{T}^n \times \Omega \to \mathbf{T}^n \times D$ given by

$$\overline{\Phi}(\theta,\omega) = (U(\theta;\omega), \nabla g^0(\phi(\omega)) + V(\theta;\omega)),$$

where U and V are obtained in Theorem 2.1. We have $H(\theta, I; \phi(\omega)) = H(\theta, \nabla g^0(\phi(\omega)) + I)$, $I \in B_R(0)$, in the notations of Sect. 2.1. Then Theorem 2.1, (i), implies that $\Lambda_\omega = \{\overline{\Phi}(\theta,\omega) : \theta \in \mathbf{T}^n\}$, $\omega \in \Omega_\kappa$, is an embedded Lagrangian invariant torus of the Hamiltonian H for each $\omega \in \Omega_\kappa$ with frequency ω. The corresponding estimates for $\overline{\Phi}$ follow directly from those in Theorem 2.1. □

Proof of Corollary 1.2. The proof is close to that of Theorem 2.1, [10] and we will be concerned mainly with the corresponding Gevrey estimates. Let $\varepsilon_H \leq \epsilon L_1^{-N-2}$. Then in the notations of (ii), Theorem 1.1, we get $AC_1 L_1^{N/2}\sqrt{\varepsilon_H} \leq AC\epsilon$. Choosing ϵ small enough and using Proposition A.2 as well as (ii), Theorem 1.1, we obtain a solution $\theta = \theta(\varphi,\omega)$ the equation $\overline{U}(\theta,\omega) = \varphi$ such that $\theta(\varphi,\omega) - \varphi$ satisfies the same Gevrey estimates as $\overline{U}$. Set $F(\varphi,\omega) = \overline{V}(\theta(\varphi,\omega),\omega)$. We have $\Lambda_\omega = \{(\varphi, F(\varphi,\omega)) : \varphi \in \mathbf{T}^n\}$ for each $\omega \in \Omega_\kappa$. Moreover, by Proposition A.4 we obtain

$$\left|\partial_\varphi^\alpha \partial_\omega^\beta (F(\varphi,\omega) - \nabla g^0(\omega))\right| \leq \kappa A C_1^{|\alpha|}(C_2\kappa^{-1})^{|\beta|}\alpha!^{\,\rho}\beta!^{\,\rho(\tau+1)+1}\, L_1^{N/2}\sqrt{\varepsilon_H}$$

uniformly in $(\theta,\omega) \in \mathbf{T}^n \times \Omega$. Hereafter, A, $C_1 = CL_1$, and $C_2 = CL_1^{\tau+1}$ are positive constants as in Theorem 1.1. Denote by $p : \mathbf{R}^n \to \mathbf{T}^n$ the natural projection. As in Lemma 2.2 , [10], we shall find $\psi \in \mathcal{G}^{\rho,\rho'}(\mathbf{R}^n \times \Omega)$ and $R \in \mathcal{G}^{\rho'}(\Omega)$ such that $Q(x,\omega) := \psi(x,\omega) - \langle x, R(\omega)\rangle$ is 2π periodic with respect to x and

(i) $\forall (x,\omega) \in \mathbf{R}^n \times \Omega_\kappa$, $\nabla_x \psi(x,\omega) = F(p(x),\omega)$,

(ii) $\left|\partial_x^\alpha \partial_\omega^\beta Q(x,\omega)\right| + \left|\partial_I^\beta (R(\omega) - \nabla g^0(\omega))\right| \leq \kappa A C_1^{|\alpha|}(C_2\kappa^{-1})^{|\beta|}\alpha!^{\,\rho}\beta!^{\,\rho(\tau+1)+1}\, L_1^{N/2}\sqrt{\varepsilon_H}$,

for $(x,\omega) \in \mathbf{R}^n \times \Omega$. To obtain ψ we consider the function

$$\widetilde{\psi}(x,\omega) = \int_{\gamma_x} \sigma = \int_0^1 \langle F(p(tx),\omega),\, x\rangle\, dt, \quad (x,\omega) \in \mathbf{R}^n \times \Omega\,,$$

where $\gamma_x = \{(tx, F(p(tx),\omega)) : 0 \leq t \leq 1\}$ and $\sigma = \xi dx$ is the canonical one-form on $T^*\mathbf{R}^n$. Then $\widetilde{\psi}(x,\omega) - \langle \nabla g^0(\omega), x\rangle$ satisfies the Gevrey estimates (ii) in $[0,4\pi]^n \times \Omega$. We set $2\pi R_j(\omega) = \widetilde{\psi}(2\pi e_j,\omega)$, $\omega \in \Omega$, $\{e_j\}$ being an unitary basis in $\mathbf{R}^n$. Then $R - \nabla g^0$ satisfies (ii) in Ω. Since Λ_ω, $\omega \in \Omega_\kappa$, is Lagrangian, we obtain as in [10] that for such ω the function $\nabla_x\widetilde{\psi}(x,\omega)$ is 2π periodic with respect to x and $\widetilde{\psi}(x+2\pi m,\omega) - \widetilde{\psi}(x,\omega) = \langle 2\pi m, R(\omega)\rangle$ for $m \in \mathbf{Z}^n$. Consider the function $\widetilde{Q}(x,\omega) = \widetilde{\psi}(x,\omega) - \langle x, R(\omega)\rangle$. It satisfies the Gevrey estimates (ii) in $[0,4\pi]^n \times \Omega$, and it is 2π periodic with respect to x for $\omega \in \Omega_\kappa$. We are going to average $\widetilde{Q}$ on $\mathbf{T}^n$. Let $f \in \mathcal{G}^\rho_C(\mathbf{R}^n)$ with $\operatorname{supp} f \subset [\pi/2, 7\pi/2]^n$, where $C > 0$ is a positive constant, and such that $\sum_{k\in\mathbf{Z}^n} f(x - 2\pi k) = 1$ for each $x \in \mathbf{R}^n$. Consider the function $Q(x,\omega) = \sum_{k\in\mathbf{Z}^n}(f\widetilde{Q})(x - 2\pi k,\omega)$. It is 2π-periodic with respect to x by construction, it belongs to $\mathcal{G}^{\rho,\rho'}(\mathbf{R}^n \times \Omega)$, and $Q(x,\omega) = \widetilde{Q}(x,\omega)$ for $(x,\omega) \in \mathbf{R}^n \times \Omega_\kappa$. Moreover, Q satisfies the Gevrey estimates (ii) in $\mathbf{R}^n \times \Omega$. We set $\psi(x,\omega) = Q(x,\omega) + \langle x, R(\omega)\rangle$. Recall that $\operatorname{dist}(\Omega_\kappa, \mathbf{R}^n \setminus \Omega) \geq \kappa$. Then multiplying Q and $R - \nabla g^0$ by a suitable cut-off function $h \in \mathcal{G}^{\rho'}_{\widetilde{C}}(\Omega)$, with $\widetilde{C} = C\kappa^{-1}$ and $C > 0$ independent of $\Omega \subset \Omega_0$, such that $h = 1$ in a neighborhood of Ω_κ and $h(\omega) = 0$ if $\operatorname{dist}(\omega, \mathbf{R}^n \setminus \Omega) \leq \kappa/2$, we can assume that $\psi(x,\omega) = \langle x, \nabla g^0(\omega)\rangle$ for any ω such that $\operatorname{dist}(\omega, \mathbf{R}^n \setminus \Omega) \leq \kappa/2$. This does not change the corresponding Gevrey estimates for ψ.

Let $\varepsilon_H L_1^{N+2(\tau+2)} \leq \epsilon \ll 1$. Then $\kappa A C_1 (C_2\kappa^{-1}) L_1^{N/2}\sqrt{\varepsilon_H} \leq AC^2\epsilon \ll 1$, and the map $\Omega \ni \omega \to \nabla_x\psi(x,\omega) \in D$ becomes a diffeomorphism for any x fixed, which gives a $\mathcal{G}^{\rho,\rho'}$-foliation of $\mathbf{T}^n \times D$ by Lagrangian tori $\Lambda_\omega = \{(p(x), \nabla_x\psi(x,\omega)) : x \in \mathbf{R}^n\}$, $\omega \in \Omega$. The action $I = (I_1, \dots, I_n) \in D$ on each Λ_ω, $\omega \in D$, is given by

$$I_j(\omega) = (2\pi)^{-1} \int_{\gamma_j(\omega)} \sigma = (2\pi)^{-1}(\psi(2\pi e_j, \omega) - \psi(0,\omega)) = R_j(\omega),$$

where $\gamma_j(\omega) = \{(p(te_j), \nabla_x\psi(te_j,\omega)) : 0 \le t \le 2\pi\}$. Then $I(\omega) - \nabla g^0(\omega) = R(\omega) - \nabla g^0(\omega)$ satisfies (ii) in Ω, and choosing $\varepsilon_H L_1^{N+2(\tau+2)} \le \epsilon \ll 1$ we obtain that the frequency map $\Omega \ni \omega \mapsto I(\omega) \in D$ is a diffeomorphism of Gevrey class $\mathcal{G}^{\rho'}$. Using Remark A.1 we show that the inverse map $D \ni I \mapsto \omega(I) \in \Omega$ is in $\mathcal{G}^{\rho'}$ and

$$\left|\partial^\alpha \left(\omega(I) - \nabla H^0(I)\right)\right| \le \kappa A \,(C_2\kappa^{-1})^{|\alpha|}\, \alpha!^{\,\rho'} L_1^{N/2}\sqrt{\varepsilon_H}\,,$$

uniformly with respect to $(\varphi,\omega) \in \mathbf{T}^n \times \Omega$ and for any $\alpha, \beta \in \mathbf{N}^n$. Now we set $\Phi(x,I) = \psi(x,\omega(I))$. Then $\Phi(x,I) - \langle x, I\rangle$ is 2π-periodic with respect to x, and using Proposition A.3 we get the estimates

$$\left|\partial_x^\alpha \partial_I^\beta (\Phi(x,I) - \langle x, I\rangle)\right| \le \kappa A C_1^{|\alpha|} (C_2\kappa^{-1})^{|\beta|} \alpha!^{\,\rho} \beta!^{\,\rho(\tau+1)+1} L_1^{N/2}\sqrt{\varepsilon_H}.$$

Solving the equation $\Phi_I(\theta, I) = \varphi$ with respect to θ by means of Proposition A.2 we obtain the symplectic transformation χ. For any $\omega \in \Omega_\kappa$ and any θ we have $(\theta, F(\theta,\omega)) = (\theta, \Phi_\theta(\theta, I(\omega))) = \chi(\Phi_I(\theta, I(\omega)), I(\omega))$, hence, $\Lambda_\omega = \chi(\mathbf{T}^n \times \{I(\omega)\})$. Set $\tilde H(\varphi, I) = H(\chi(\varphi, I))$. Then $\tilde H$ is constant on $\mathbf{T}^n \times \{I(\omega)\}$. We set $K(I) = \tilde H(0, I)$ and $R(\varphi, I) = \tilde H(\varphi, I) - K(I)$. Then $R(\varphi, I) = 0$ on $\mathbf{T}^n \times E_\kappa$, hence, all the derivatives of R vanish on $\mathbf{T}^n \times E_\kappa$, since each point of E_κ is of positive Lebesgue density in E_κ. Using Proposition A.4 for $\tilde H(\varphi, I) = H(\theta(\varphi, I), \Phi_\theta(\theta(\varphi, I), I))$ we obtain for any $\alpha, \beta \in \mathbf{N}^n$

$$\left|\partial_\varphi^\alpha \partial_I^\beta (\tilde H(\varphi, I) - H^0(I))\right| \le \kappa A C_1^{|\alpha|} (C_2\kappa^{-1})^{|\beta|} \alpha!^{\,\rho} \beta!^{\,\rho(\tau+1)+1} L_1^{N/2}\sqrt{\varepsilon_H}$$

uniformly with respect to $(\varphi, I) \in \mathbf{T}^n \times D$, where the constants A, C_1 and C_2 are as above. □

Appendix

We shall obtain a variant of the implicit function theorem of Komatsu [3] in anisotropic Gevrey classes. Let X and Ω_0 be domains in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectevly. Fix $\rho' \ge \rho \ge 1$. Let $F = (f_1, \dots, f_n)$ be a Gevrey function of the class $\mathcal{G}^{\rho,\rho'}(X \times \Omega, \mathbf{R}^n)$. We suppose that there are constants $A_0 > 0$ and $h_1, h_2 > 0$, and a small parameter $0 < \varepsilon \le 1$, such that

$$\forall (x,\omega) \in X \times \Omega_0\,, \quad \left|\partial_x^\alpha \partial_\omega^\beta (F(x,\omega) - x)\right| \le \varepsilon A_0 h_1^{|\alpha|} h_2^{|\beta|} \alpha!^{\,\rho} \beta!^{\,\rho'} \tag{A.1}$$

for any multi-indices α, β. We choose $\varepsilon > 0$ so that $\varepsilon A_0 h_1 \le 1/2$. Then $|D_x F(x,\omega) - \mathrm{Id}| \le 1/2$ for any $(x,\omega) \in X \times \Omega_0$, where $D_x F$ stands for the Jacobian of F with respect to x. Hence, the inverse matrix $(D_x F)^{-1}$ exists and $\left|D_x F(x,\omega)^{-1}\right| \le 2$ in $X \times \Omega_0$. Consider now a local solution $x = g(y,\omega)$, $(y,\omega) \in Y \times \Omega$ of $F(x,\omega) = y$, where $Y \subset \mathbf{R}^n$ and $\Omega \subset \Omega_0$ are suitable domains.

Proposition A. 1 *Suppose that (A.1) holds and with* $0 < \varepsilon \le 1$ *and* $\varepsilon A_0 h_1 \le 1/2$, *and consider a local solution* $x = g(y,\omega)$, $(y,\omega) \in Y \times \Omega$, *of* $F(x,\omega) = y$. *Then* $g \in \mathcal{G}^{\rho,\rho'}(Y \times \Omega, X)$, *and there exist positive constants* A *and* C *depending only on* ρ, ρ', n, *and* m, *such that*

$$\forall\, \alpha, \beta, \quad \sup_{Y\times\Omega} \left|\partial_y^\alpha \partial_\omega^\beta (g(y,\omega) - y)\right| \le \varepsilon A_0 A C^{|\alpha+\beta|}\, h_1^{|\alpha|}\, h_2^{|\beta|}\, \alpha!^{\,\rho} \beta!^{\,\rho'} .$$

Remark A.1. Let $f = (f_1, \ldots, f_n) \in \mathcal{G}^\rho(X, \mathbf{R}^n)$ and $|\partial_x^\alpha (f(x) - x)| \le \varepsilon A_0 h^{|\alpha|} \alpha!^{\,\rho}$ in X for any α, where $0 < \varepsilon \le 1$ and $\varepsilon A_0 h \le 1/2$, and let $g : Y \to X$ be an inverse map to f. Then the inverse map g of f belongs to $\mathcal{G}^\rho(Y, X)$ and $\left|\partial_y^\alpha (g(y) - y)\right| \le \varepsilon A_0 A C^{|\alpha|}\, h^{|\alpha|}\, \alpha!^{\,\rho}$ in Y for any α.

As a corollary we obtain

Proposition A. 2 *Suppose that* $F \in \mathcal{G}^{\rho,\rho'}(\mathbf{T}^n \times \Omega, \mathbf{T}^n)$ *satisfies (A.1) with* $0 < \varepsilon \le 1$ *and* $\varepsilon A_0 h_1 \le 1/2$. *Then there exists* $g \in \mathcal{G}^{\rho,\rho'}(\mathbf{T}^n \times \Omega, \mathbf{T}^n)$ *and there are positive constants* A *and* C *depending only on* ρ, ρ', n, *and* m *such that*

$$\forall\, \alpha, \beta, \quad \sup_{X\times\Omega} \left|\partial_y^\alpha \partial_\omega^\beta (g(y,\omega) - y)\right| \le \varepsilon A_0 A C^{|\alpha+\beta|} h_1^{|\alpha|}\, h_2^{|\beta|}\, \alpha!^{\,\rho} \beta!^{\,\rho'} .$$

Proof of Proposition A 1. We rescale the variables

$$x \mapsto h_1 x = \tilde{x} \in h_1 X := \tilde{X}\,, \quad y \mapsto h_1 y = \tilde{y} \in h_1 Y := \tilde{Y}\,, \quad \omega \mapsto h_2 \omega = \tilde{\omega} \in h_2 \Omega := \tilde{\Omega}\,.$$

Set $\tilde{F}(x,\omega) = h_1 F(h_1^{-1} x, h_2^{-1}\omega)$, $(x,\omega) \in \tilde{X} \times \tilde{\Omega}$, and $\tilde{g}(y,\omega) = h_1 g(h_1^{-1} y, h_2^{-1}\omega)$, $(y,\omega) \in \tilde{Y} \times \tilde{\Omega}$. Then $x = \tilde{g}(y,\omega)$, $(y,\omega) \in \tilde{Y} \times \tilde{\Omega}$, is a solution of the equation $\tilde{F}(x,\omega) = y$. Moreover, (A.1) implies

$$\sup_{\tilde{X}\times\tilde{\Omega}} \left|\partial_x^\alpha \partial_\omega^\beta (\tilde{F}(x,\omega) - x)\right| \le \tilde{\varepsilon} \alpha!^{\,\rho} \beta!^{\,\rho'}, \tag{A.2}$$

where $\tilde{\varepsilon} := \varepsilon A_0 h_1 \le 1/2$. We are going to prove that there exist positive constants A and C depending only on ρ, ρ', n, and m such that

$$\forall\, \alpha, \beta, \quad \sup_{\tilde{Y}\times\tilde{\Omega}} \left|\partial_y^\alpha \partial_\omega^\beta (\tilde{g}(y,\omega) - y)\right| \le \tilde{\varepsilon} A C^{|\alpha+\beta|}\, \alpha!^{\,\rho} \beta!^{\,\rho'} . \tag{A.3}$$

Then rescaling back the variables we obtain the estimates in Proposition A.1.

We are going to prove (A.3). From now on we omit '$\sim$' to simplify the notations. The implicit function theorem of Komatsu [3] implies that $g \in \mathcal{G}^{\rho'}(Y \times \Omega; X)$. Moreover, it follows from [3] that

$$\sup_{X\times\Omega} \left|\partial_x^\alpha \partial_\omega^\beta (g_j(y,\omega) - y_j)\right| \le \varepsilon A\, h^{|\alpha|+|\beta|} (\alpha!\, \beta!)^{\,\rho'} . \tag{A.4}$$

Hereafter, A and h are positive constants, depending only on ρ, ρ', n, and m. To obtain (A.4) we consider the inverse mapping (g, id) of (F, id) in $X \times \Omega$ and we use that $B = 2$, $0 < C = \varepsilon \le 1$ and $h = 1$ in the estimates of b_r in (4), [3], p. 70-71 (see also the estimates of $\psi^{r,\alpha}$ below).

Take any $x^0, y^0 \in X$ and $\omega^0 \in \Omega$ such that $F(x^0, \omega^0) = y^0$. Changing the variables if necessary we assume $y^0 = 0$ and $\omega^0 = 0$. Consider now the solution $\omega \mapsto g(0,\omega)$ of $F(x,\omega) = 0$ with $g(0,0) = x^0$. Then we obtain

$$F(x,\omega) = (x - g(0,\omega)) D_x F(g(0,\omega),\omega) + R(x,\omega),$$

in a neighborhood of $(0,0)$, where $R(x,\omega)=O(|x-g(0,\omega)|^2)$, $R\in\mathcal{G}^{\rho,\rho'}(X\times\Omega;\mathbf{R}^n)$. Moreover, it follows from (A.2), (A.4) and from Proposition A.3 below that

$$\sup_{X\times\Omega}\left|\partial_x^\alpha\partial_\omega^\beta R(x,\omega)\right|\le\varepsilon Ah^{|\alpha+\beta|}\alpha!^{\,\rho}\beta!^{\,\rho'},$$

where $A>0$ and $h>0$ depend only on ρ, ρ', n, and n. Now me make a local change of the variables

$$z=(x-g(0,\omega))D_xF(g(0,\omega),\omega),$$

and we obtain the equation (with respect to z)

$$z=y+\varphi(z,\omega)\,,\quad z\in Y_0,\,y\in Y,\,\omega\in\Omega\,,\tag{A.5}$$

where Y_0 and Y are neighborhoods of 0 in $\mathbf{R}^n$, $\varphi(0,\omega)\equiv 0$, and

$$\forall\,\alpha,\beta,\quad\sup_{Y_0\times\Omega}\left|\partial_z^\alpha\partial_\omega^\beta\varphi(z,\omega)\right|\le\varepsilon Ah^{|\alpha+\beta|}\alpha!^{\,\rho}\beta!^{\,\rho'}$$

with some positive constants A and h as above. We are going to estimate the derivatives $\partial_y^p\partial_\omega^\alpha u(0,0)$ of the solution $z=u(y,\omega)$ of the equation above.

For $j\in\mathbf{N}$ we set $M_j=j!^{\,\rho}$ and $N_j=j!^{\,\rho'}$. Then $M_j\le N_j$ and there is $H\ge 1$ such that

$$\left(M_q/q!\right)^{1/q-1}\le H\left(M_p/p!\right)^{1/p-1},\ \left(M_q/q!\right)^{1/q}\le H\left(M_p/p!\right)^{1/p},\ 2\le q\le p.\tag{A.6}$$

The same estimates hold for N_j as well.

Consider the Taylor expansion of φ in formal power series at $(0,0)$

$$\varphi(z,\omega)=\sum_{|p|\ge 2}\sum_{\alpha\in\mathbf{N}^m}\frac{\varphi^{p,\alpha}}{p!\alpha!}z^p\omega^\alpha\,,\quad\varphi^{p,\alpha}=\partial_z^p\partial_\omega^\alpha\varphi(0,0).$$

Then we have

$$\varphi\ll\varepsilon A\sum_{|p|\ge 2}\sum_{\alpha\in\mathbf{N}^m}\frac{M_{|p|}}{p!}\frac{N_{|\alpha|}}{\alpha!}h^{|p|+|\alpha|}z^p\omega^\alpha,$$

which means that $|\varphi^{p,\alpha}|\le\varepsilon AM_{|p|}N_{|\alpha|}h^{|p|+|\alpha|}$ for each p,α. Consider the solution $z=u(y,\omega)$ of (A.5) and its formal Taylor series at $(0,0)$

$$u(y,\omega)=y+\sum_{|p|\ge 2}\sum_{\alpha\in\mathbf{N}^m}\frac{u^{p,\alpha}}{p!\alpha!}y^p\omega^\alpha.$$

Then (A.5) is formally equivalent to

$$\sum_{|p|\ge 2}\sum_{\alpha\in\mathbf{N}^m}\frac{u^{p,\alpha}}{p!\alpha!}y^p\omega^\alpha=\sum_{|p|\ge 2}\sum_{\alpha\in\mathbf{N}^m}\frac{\varphi^{p,\alpha}}{p!\alpha!}\left(y+\sum_{|q|\ge 2}\sum_{\beta\in\mathbf{N}^m}\frac{u^{q,\beta}}{q!\beta!}y^q\omega^\beta\right)^p\omega^\alpha.$$

Denote by $v(y,\omega)$ the power series

$$v(y,\omega)=\sum_{|p|\ge 2}\sum_{\alpha\in\mathbf{N}^m}\frac{v^{p,\alpha}}{p!\alpha!}y^p\omega^\alpha,$$

and suppose that it is a formal solution of

$$v(y,\omega)=\varepsilon A\sum_{|p|\geq 2}\sum_{\alpha\in\mathbf{N}^m}\frac{M_{|p|}}{p!}\frac{N_{|\alpha|}}{\alpha!}\,h^{|p|}h^{|\alpha|}\,(y+v(y,\omega))^p\,\omega^\alpha\,. \tag{A.7}$$

We claim that such a solution exists and it is unique, $v^{p,\alpha}>0$, and $u\ll y+v$. Indeed, for any $|p|\geq 2$, $\alpha\in\mathbf{N}^m$, and $1\leq j\leq n$, there is a polynomial $Q_{p,\alpha,j}\not\equiv 0$ of the variables $(t_{q,\beta},s_{r,\gamma})$ and with non-negative coefficients, where $q,r\in\mathbf{N}^n$, $\beta,\gamma\in\mathbf{N}^m$, $2\leq|q|\leq|p|$, $\beta\leq\alpha$, $2\leq|r|\leq|p|-1$, $\gamma\leq\alpha$, and such that $u_j^{p,\alpha}$, respectively, $v_j^{p,\alpha}$, is the value of $Q_{p,\alpha,j}$ for $t_{q,\beta}=\varphi^{q,\beta}$, $s_{r,\gamma}=u_j^{r,\gamma}$, respectively, for $t_{q,\beta}=\varepsilon AM_{|p|}N_{|\alpha|}h^{|p|+|\alpha|}$, $s_{r,\gamma}=v_j^{r,\gamma}$. For $|p|=2$ the polynomial $Q_{p,\alpha,j}$ is independent of $s_{r,\gamma}$, $|r|\geq 2$. Since the coefficients of $Q_{p,\alpha,j}$ are non-negative, we get $|u_j^{r,\alpha}|\leq v_j^{r,\alpha}$ for $|r|=2$ and each α. By recurrence we obtain $|u_j^{r,\alpha}|\leq v_j^{r,\alpha}$ for each $|r|\geq 2$ and $\alpha\in\mathbf{N}^m$.

Now we set $y=(s,\dots,s)$, $s>0$, and consider

$$\phi(s,\omega)=\sum_{r=2}^{\infty}\sum_{\alpha\in\mathbf{N}^m}\frac{\phi^{r,\alpha}}{r!\alpha!}s^r\omega^\alpha,\ \ \phi^{r,\alpha}=\sum_{|p|=r}v^{p,\alpha}\frac{r!}{p!}.$$

Note that for such y the right hand side of the equation (A.7) is invariant under any permutation of the components $v_j(y,\omega)$, $1\leq j\leq n$, and using the uniqueness of the solution, we obtain $\phi_1(s,\omega)=\cdots=\phi_n(s,\omega)=\tilde\phi(s,\omega)$. Then $t=s+\tilde\phi(s,\omega)$ is a formal solution of

$$t=s+\varepsilon A\sum_{r=2}^{\infty}\sum_{\alpha\in\mathbf{N}^m}\sum_{|p|=r}\frac{r!}{p!}\frac{M_r}{r!}\frac{N_{|\alpha|}}{\alpha!}(ht)^r(h\omega)^\alpha\,,$$

where $\sum_{|p|=r}\frac{r!}{p!}\leq n^r$. Let $t=\psi(s,\omega):=s+\sum_{r=2}^{\infty}\sum_{\alpha\in\mathbf{N}^m}\frac{\psi^{r,\alpha}}{r!\alpha!}s^r\omega^\alpha$ be the formal solution of

$$t=s+\varepsilon A\sum_{r=2}^{\infty}\sum_{\alpha\in\mathbf{N}^m}\frac{M_r}{r!}\frac{N_{|\alpha|}}{\alpha!}(hnt)^r(h\omega)^\alpha\,. \tag{A.8}$$

We obtain as above $s+\tilde\phi\ll\psi$, which implies $|u_j^{p,\alpha}|\leq\psi^{|p|,\alpha}$ for $1\leq j\leq n$ and for any $|p|\geq 2$, $\alpha\in\mathbf{N}^m$.

We are going to estimate $\psi^{r,\alpha}$. Set $C(\omega)=A\sum_{\alpha\in\mathbf{N}^m}\frac{N_{|\alpha|}}{\alpha!}(h\omega)^\alpha$. Then (A.8) becomes

$$s=k(t,\omega):=t-\varepsilon C(\omega)\sum_{r=2}^{\infty}\frac{M_r}{r!}(hnt))^r.$$

Recall that $0<\varepsilon\leq 1$. We suppose also that $A\geq 1$ and $h\geq 1$. As in [3], using the Lagrange expansion theorem, we obtain for $r\geq 2$

$$\begin{aligned}\psi^{r,\alpha}\ &=\ \left[\partial_\omega^\alpha\left\{\left(\frac{d}{dt}\right)^{r-1}\left(\frac{t}{k(t,\omega)}\right)^r\right\}|_{t=0}\right]|_{\omega=0}\\ &\leq\ \varepsilon\left[\partial_\omega^\alpha\left\{\left(\frac{d}{dt}\right)^{r-1}\left(\sum_{q=0}^{\infty}\left(C(\omega)hn\sum_{p=1}^{r-1}\frac{M_{p+1}}{(p+1)!}(hnt)^p\right)^q\right)^r\right\}|_{t=0}\right]|_{\omega=0}\,,\end{aligned}$$

and we get

$$\begin{aligned}
\psi^{r,\alpha} &\le \varepsilon M_r\,(4Hhn)^{r-1}\left[\partial_\omega^\alpha\left(1+Ahn\sum_{\beta\le\alpha}\frac{N_{|\beta|}}{|\beta|!}(h\omega)^\beta\right)^{r-1}\right]|_{\omega=0}\\
&\le \varepsilon M_r(8AHh^2n^2)^{r-1}\left[\partial_\omega^\alpha\left(\sum_\beta\left(\left(\frac{N_{|\alpha|}}{|\alpha|!}\right)^{1/|\alpha|}h\omega\right)^\beta\right)^{r-1}\right]|_{\omega=0}\\
&\le \varepsilon M_r N_{|\alpha|}(8AHh^2n^2)^{r-1}h^{|\alpha|}2^{m(r-2)+|\alpha|}.
\end{aligned}$$

We have used the inequalities (A.6) for M_j and N_j, as well as the identity

$$\left(\sum_{\beta\in\mathbf{N}^m}\omega^\beta\right)^q=\sum_{\alpha\in\mathbf{N}^m}C_\alpha\omega^\alpha,$$

where $q=r-1\ge 1$ and

$$C_\alpha=\begin{pmatrix} q+\alpha_1-1\\ \alpha_1\end{pmatrix}\cdots\begin{pmatrix} q+\alpha_m-1\\ \alpha_m\end{pmatrix}\;\le\;2^{m(q-1)+|\alpha|}.$$

Thus we obtain

$$\forall\,(p,\alpha,j),\quad |u_j^{p,\alpha}|\le\psi^{|p|,\alpha}\le\varepsilon A\,M_{|p|}N_{|\alpha|}h^{r+|\alpha|}$$

where $A,h>0$ depend only on ρ, ρ', n, and m. Changing back the variables and rescaling back Ω, we obtain the desired estimates for g.

Finally we recall the Gevrey estimates for the composition of two functions in anisotropic Gevrey classes.

Proposition A. 3 *Let $g\in\mathcal{G}^\mu_{C_1}(\Omega,Y)$ and $f\in\mathcal{G}^{\rho,\mu}_{B,C_2}(X\times Y)$, where $\mu\ge\rho\ge1$, B, C_1 and C_2 are positive constants and X, Y and Ω are open sets in $\mathbf{R}^n$. Suppose that $\|g\|_{C_1}=A_1$ and $\|f\|_{B,C_2}=A_2$ in the corresponding Gevrey norms with A_1, $A_2>0$. Then the composition $(x,\omega)\mapsto F(x,\omega):=f(x,g(\omega))$ belongs to $\mathcal{G}^{\rho,\mu}_{B,C}(X\times\Omega)$, where $C=2^{n+\mu}n^\mu C_1\max(1,A_1C_2)$ and $\|F\|_C\le A_2$.*

Proof. Using the Faa de Bruno formula we write

$$\partial_x^\gamma\partial_\omega^\alpha F(x,\omega)=\sum_{1\le|\beta|=p\le|\alpha|}\frac{(\partial_x^\gamma\partial_y^\beta f)(x,g(\omega))}{\beta!}\sum_{\substack{\alpha^1+\cdots+\alpha^p=\alpha\\ |\alpha^1|\ge1,\dots,|\alpha^p|\ge1}}\frac{\alpha!}{\alpha^1!\cdots\alpha^p!}\,(\partial_\omega^{\alpha^1}g)(\omega)\cdots(\partial_\omega^{\alpha^p}g)(\omega).$$

Since $|\partial_\omega^\alpha g(\omega)|\le A_1C_1^{|\alpha|}\alpha!^\mu$, and $j!\le 2^j(j-1)!$ for $j\ge1$, we estimate above $|\partial_x^\gamma\partial_\omega^\alpha F(x,\omega)|$ by

$$A_2B^{|\gamma|}(2^{\mu-1}C_1)^{|\alpha|}\gamma!^{\,\rho}\,\alpha!\sum_{1\le|\beta|=p\le|\alpha|}\;\sum_{\substack{\alpha^1+\cdots+\alpha^p=\alpha\\ |\alpha^1|\ge1,\dots,|\alpha^p|\ge1}}(A_1C_2)^p\;\Big((|\alpha^1|-1)!\cdots(|\alpha^p|-1)!p!\Big)^{\mu-1}$$

We have $(|\alpha^1|-1)!\cdots(|\alpha^p|-1)!p!\le|\alpha|!\le n^{|\alpha|}\alpha!$. Observe that for any $r,p\in\mathbf{N}$, $r,p\ge1$, the number of multi-indices $\gamma=(\gamma_1,\dots,\gamma_p)\in\mathbf{N}^p$ with $|\gamma|=r$ is given by $\begin{pmatrix} r+p-1\\ p-1\end{pmatrix}$ (see [11], Sect. 1.2). Then the number of partitions $\alpha^1+\dots+\alpha^p=\alpha$, $\alpha^j\in\mathbf{N}^n$, of $\alpha=(\alpha_1,\dots,\alpha_n)$ is given by

$$\binom{\alpha_1+p-1}{p-1}\cdots\binom{\alpha_n+p-1}{p-1}\ \le\ 2^{|\alpha|}2^{n(p-1)},$$

which implies

$$|\partial_x^\gamma\partial_\omega^\alpha F(x)|\le A_2 2^{-n}B^{|\gamma|}\left(2^{n+\mu}n^\mu C_1\max(1,A_1C_2)\right)^{|\alpha|}\gamma!^{\,\rho}\,\alpha!^{\,\mu}\sum_{\beta\in\mathbf{N}^n}2^{-n|\beta|}<A_2B^{|\gamma|}C^{|\alpha|}\gamma!^{\,\rho}\,\alpha!^{\,\mu}.$$

In the same way we obtain

Proposition A. 4 *Let $g\in\mathcal{G}^{\rho,\mu}_{B_1,C_1}(X\times\Omega,Y)$ and $f\in\mathcal{G}^{\rho,\mu}_{B_2,C_2}(Y\times\Omega)$, where $\mu\ge\rho\ge 1$, B_j and C_j, $j=1,2$, are positive constants, and X, Y, and Ω are open sets in $\mathbf{R}^n$. Suppose that $\|g\|_{B_1,C_1}=A_1$ and $\|f\|_{B_2,C_2}=A_2$ in the corresponding Gevrey norms with A_1, $A_2>0$. Then the composition $(x,\omega)\mapsto F(x,\omega):=f(g(x,\omega),\omega)$ belongs to $\mathcal{G}^{\rho,\mu}_{B,C}(X\times\Omega)$, where $B=2^{n+\rho}(2n)^\rho B_1\max(1,A_1B_2)$, $C=C_2+2^{n+\rho}(2n)^\rho C_1\max(1,A_1B_2)$, and $\|F\|_{B,C}\le A_2$.*

To prove the Gevrey estimates we set $z=(x,\omega)$ and using Leibnitz formula and the Faa de Bruno formula we write

$$\begin{aligned}\partial_x^\gamma\partial_\omega^\alpha F(x,\omega)\ &=\ \sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}\sum_{1\le|\beta|=p\le|\gamma+\alpha'|}\frac{(\partial_y^\beta\partial_\eta^{\alpha-\alpha'}f)(g(z),\eta)|_{\eta=\omega}}{\beta!}\\ &\times\sum_{\substack{\delta^1+\dots+\delta^p=\delta=(\gamma,\alpha')\\ |\delta^1|\ge1,\dots,|\delta^p|\ge1}}\frac{\delta!}{\delta^1!\cdots\delta^p!}\,(\partial_z^{\delta^1}g)(z)\cdots(\partial_z^{\alpha^p}g)(z),\end{aligned}$$

where $\delta^j=(\gamma^j,\alpha^j)$. On the other hand,

$$\begin{aligned}&(\gamma^1!\cdots\gamma^p!)^{\rho-1}(\alpha^1!\cdots\alpha^p!)^{\mu-1}\beta!^{\,\rho-1}\\ &\le 2^{(\rho-1)|\gamma+\alpha'|}\left((|\gamma^1+\alpha^1|-1)!\cdots(|\gamma^p+\alpha^p|-1)!p!\right)^{\rho-1}(\alpha^1!\cdots\alpha^p!)^{\mu-\rho}\\ &\le 2^{(\rho-1)|\gamma+\alpha'|}(2n)^{(\rho-1)|\gamma+\alpha'|}\gamma^\rho\alpha'!^\mu\end{aligned}$$

and as above we complete the proof of the proposition.

Proof of Lemma 3.4. First, using the Cauchy formula, we obtain as in [9], Lemma A.3, that $f:O_{2(\tilde{\upsilon}+\upsilon)h}\to\mathbf{C}$ is one-to-one (injective). Note that $2(\tilde{\upsilon}+\upsilon)=1-4\upsilon$. To show that $O_{\tilde{\upsilon}h}\subset f(O_{(1-4\upsilon)h})$, we take $w\in O_{\tilde{\upsilon}h}$ and $\omega\in\Omega_\kappa$ such that $|w-\omega|<\tilde{\upsilon}h$. Set $f=\mathrm{id}-F$. Then $|F|_h\le\upsilon h$ and $|DF|_{(1-4\upsilon)h}\le 1/4$ by the Cauchy estimates. Put $u_0=\omega$ and $u_{k+1}=F(u_k)+w$ for $k\ge 0$. By recurrence we prove

$$|u_{k+1}-u_k|\le 2^{-1}4^{-k}(1-4\upsilon)h\ ,\quad |u_k-\omega|\le(2/3)(1-4^{-k})(1-4\upsilon)h.$$

Taking the limit we find $u\in B_r(\omega)$, $r=(1-4\upsilon)h$, such that $u=F(u)+w$. The corresponding estimates of ϕ follow from the arguments in [9], Lemma A.3. □

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